A descriptive set-theoretic view of classification problems in operator algebras

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In this talk, I will attempt to give an overview of how the abstract theory of definable equivalence relations, which has been developed in descriptive set theory in past 25 years, can be used to obtain non-trivial information about the complexity of classification problems that occur in the field of operator algebras.

Operator algebras is a vast field within functional analysis which studies (bounded) operators on (mostly separable) Hilbert spaces. The field originated in the 1930s and grew out of the mathematical needs of quantum mechanics and statistical mechanics, though there are also intrinsic mathematical motivations behind the field. In many situations, it turns out to be more fruitful to study "rings of operators", or "algebras of operators", rather than single operators. Operator algebras are traditionally divided into two separate subfields: Von Neumann algebras, which form a non-commutative analogue of measure theory, and C^*-algebras, which form a non-commutative analogue of point set topology.

In both areas, classification problems play an important role: In von Neumann algebra theory, the classification of "factors", which are the building blocks of von Neumann algebras, is considered central. In C^*-algebras, the attempt to classify all amenable, simple, separable C^*-algebras is the aim of the famous "Elliott programme".

I will explain how all these classification problems can be framed within classical descriptive set theory, and show how the problems then become questions about the structure of certain analytic equivalence relations in Polish spaces. By then applying various well-developed techniques from descriptive set theory, in particular Hjorth's theory of turbulent equivalence relations, we can then give both upper and lower bounds on the complexity (measured in a natural way) of the key classification problems in operator algebras. In particular, we will see that these classifications have a very high complexity indeed, and that the invariants needed to carry out a classification will have to be extremely complicated.