What does a group know?

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Based on joint works with

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- Pablo Seban
- Dmitry Shkatov
- Yi Wang
- Mike Wooldridge
- ...

Outline

- Background: group knowledge
- Generalised distributed knowledge
- Resolving distributed knowledge
- Public announcement logic with distributed knowledge
- Quantifying over group announcements
- The power of knowledge

Background: Group Knowledge

In all of the following we assume given

a finite set $N = \{1, \ldots, n\}$ of agents

a countably infinite set of primitive propositions

Epistemic logic

A model is a tuple $M = \langle W, \sim_1, \ldots, \sim_n, V \rangle$:

- W is a set of states
- $\sim_i$ is an epistemic accessibility relation
  - Sometimes assumed to be an equivalence relation (S5)
  - Sometimes assumed to be transitive, euclidian and serial (KD45)
- V is a valuation function, assigning primitive propositions to each state

Epistemic logic

Language: $\phi ::= p \mid K_i\phi \mid \neg\phi \mid \phi_1 \wedge \phi_2$

Interpretation: $(M, s) \models K_i\phi$ iff for all t s.t. $s \sim_i t$, $(M, t) \models \phi$

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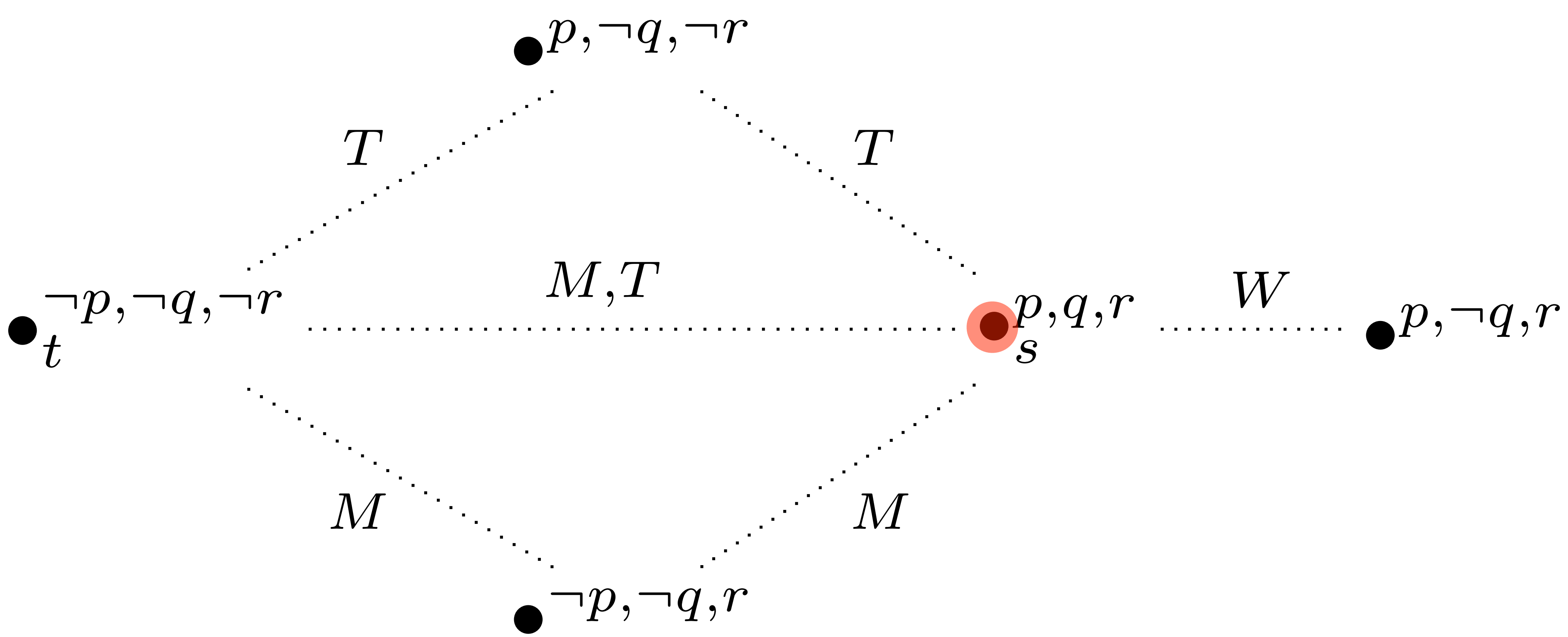

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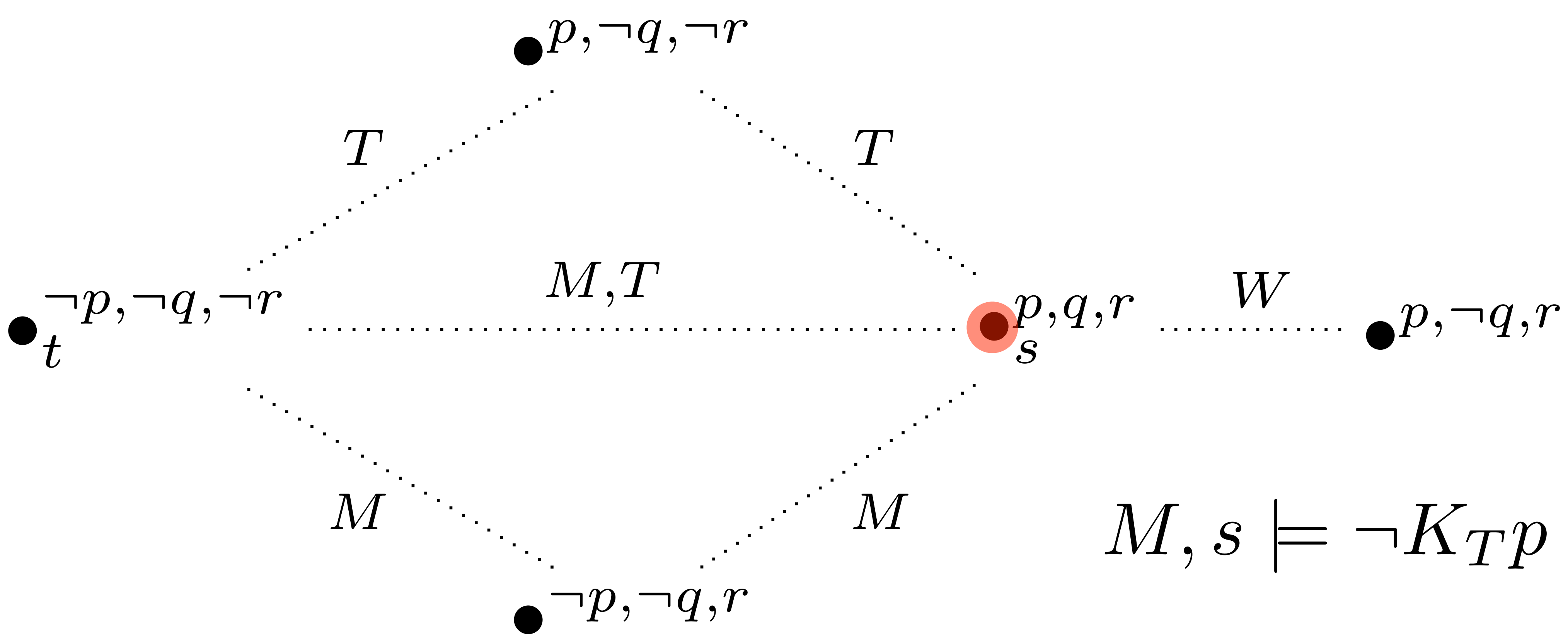

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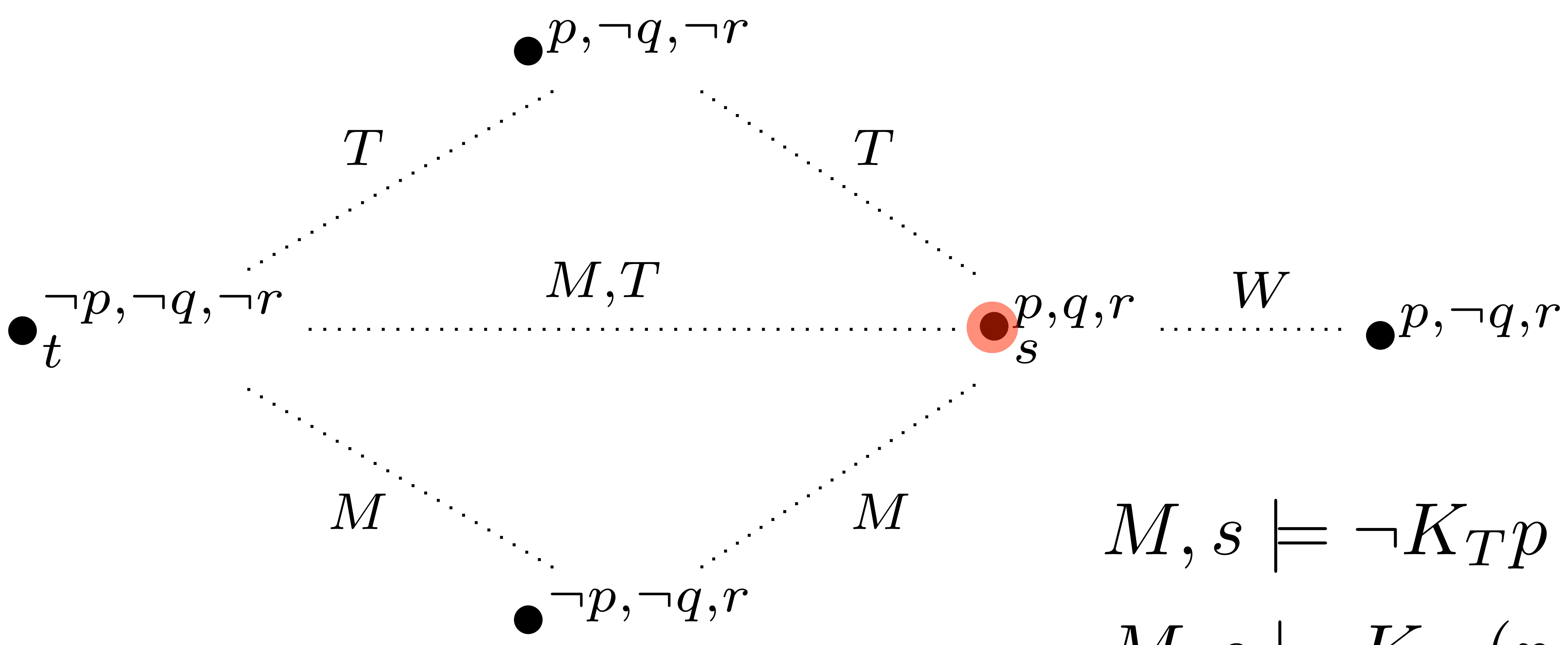

$M, s \models \neg K_T p$

$M, s \models K_M(p \rightarrow q)$

What does a group know?

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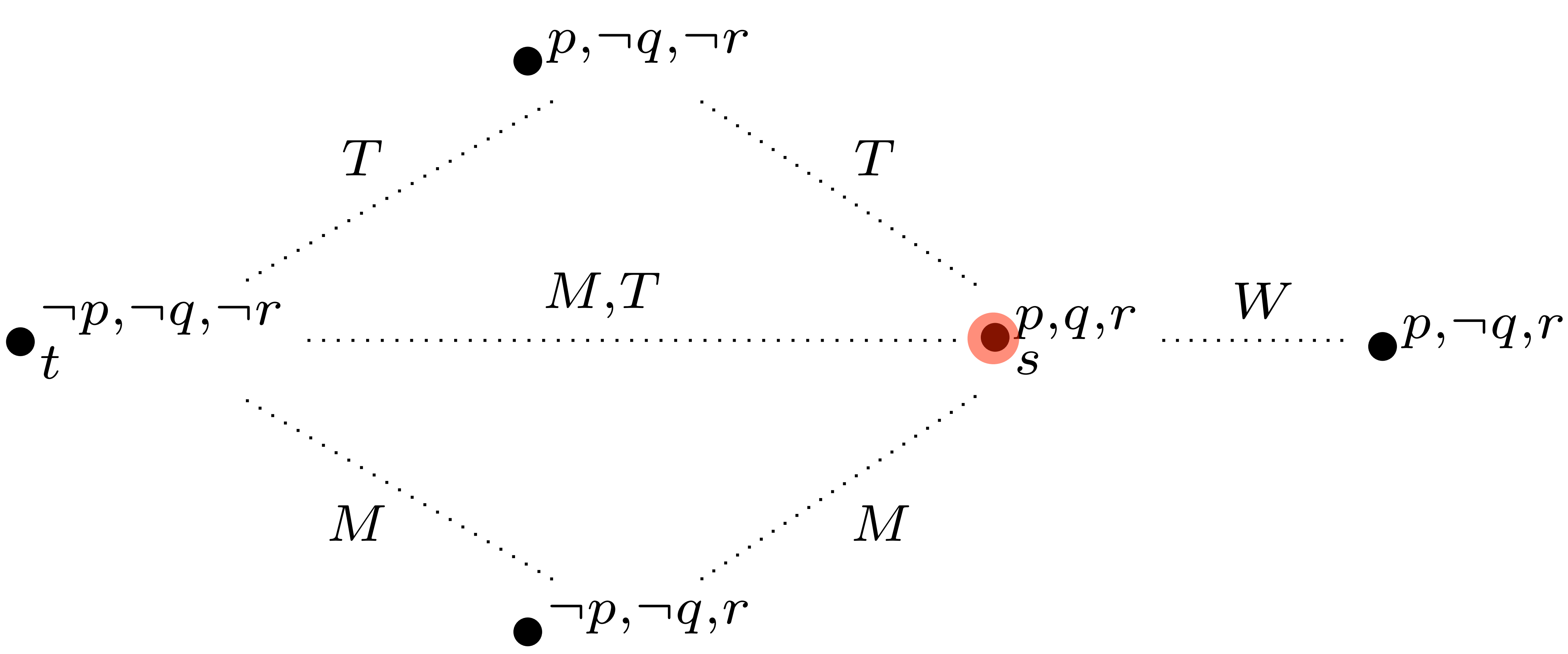

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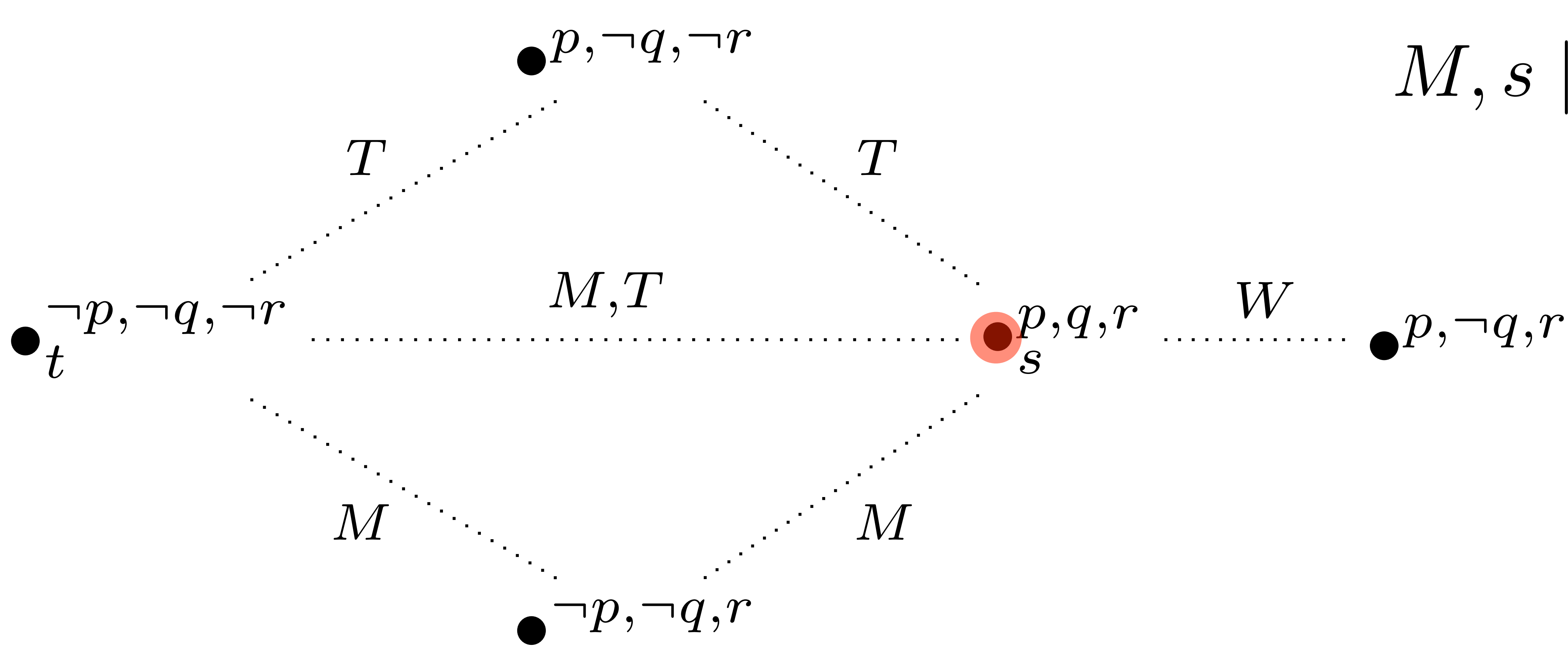

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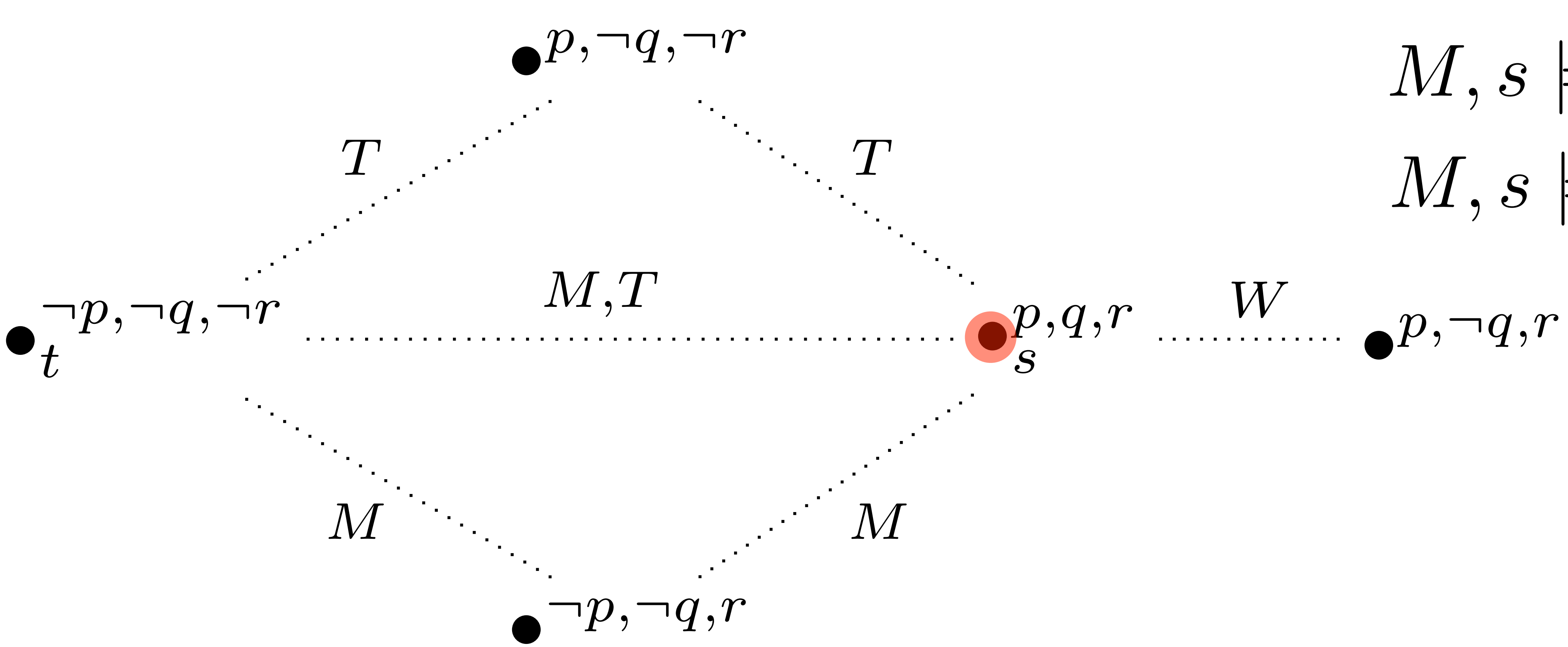

$$M, s \models D_{\{M,W\}} q$$

$$M, s \models \neg D_{\{M,T\}} q$$

Distributed Knowledge: Key Axioms

$$D_A\phi \rightarrow D_B\phi \text{ when } A \subseteq B$$

$$D_{\{a\}}\phi \leftrightarrow K_a\phi$$

Generalised Distributed Knowledge

Distributed knowledge

$$\sim_G^D = \bigcap_{i \in G} \sim_i$$

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- In other words, the group considers a state
  - impossible iff at least one member of the group considers it impossible
  - possible iff all the agents in the group considers it possible
- For S5 agents this makes sense
  - If an S5 agent considers a state impossible, then it is impossible
    - .. and this is common knowledge

Distributed knowledge for non-S5 agents

$$\sim_G^D = \bigcap_{i \in G} \sim_i$$

- The group considers a state
  - impossible iff at least one member of the group considers it impossible
  - possible iff all the agents in the group considers it possible
- For non-S5 agents, in particular agents without T/reflexivity (e.g., KD45):
  - If one agent considers a state impossible, that agent might in fact be wrong
  - Ruling out a state based on the evidence of a single agent is then a very credulous group attitude
  - Curious asymmetry between the evidence need for possibility vs. impossibility
    - impossibility: every agent is a veto voter, possibility: unanimity

Distributed knowledge for non-S5 agents

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- The group ...
  - impos... ...sible
  - possib...
- For non-... ...5):
  - If one ... ...be wrong
  - Ruling ... ...a very credul...
  - Curious asymmetry between the evidence need for possibility vs. impossibility
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Wait a moment!

Does distributed knowledge even make sense for non-S5 agents?

The KD45 properties are not closed under intersection!

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Generalised distributed knowledge (Ågotnes and Shkatov, 2014)

- In this work we look at general definitions of distributed knowledge where we vary the evidence needed for the two cases

Generalised Distributed Knowledge

- The group considers a state
  - impossible iff not at least k agents in the group considers it impossible
  - possible iff at least k agents in the group considers it possible

The generalised distributed knowledge operator

$$M, s \models D_G^{+k}\phi \Leftrightarrow \forall(s,t) \in \sim_G^{+k} M, t \models \phi$$

$$\sim_G^{+k} = \bigcup_{H \subseteq G, |H| \geq k} \bigcap_{i \in H} \sim_i$$

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$$\text{E.g., } \sim_G^{maj} = \sim_G^{+\lceil (|G|+1)/2 \rceil}$$

Expressive power and succinctness

$$\phi ::= p \mid \neg\phi \mid \phi \land \phi \mid K_i\phi \mid D_G\phi \mid D_G^{+k}\phi$$

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$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid D_G\phi \mid D_G^{+k}\phi$$

$$(M, s) \models D_G^{+k}\phi \Leftrightarrow (M, s) \models \bigwedge_{H \subseteq G, |H| \geq k} D_H\phi$$

Generalised distributed knowledge in Epistemic Logic with Quantification over Coalitions

- Epistemic Logic with Quantification over Coalitions (Ågotnes, van der Hoek and Wooldridge, 2008) use *coalition predicates* to allow more succinct epistemic expressions

- Can express generalised distributed knowledge succinctly:

$$(M, s) \models D_G^{+k} \phi \Leftrightarrow (M, s) \models \bigwedge_{H \subseteq G, |H| \geq k} D_H \phi$$

$$\Leftrightarrow (M, s) \models [geq(k) \wedge subseteq(G)]_D \phi$$

Generalised distributed knowledge

- Not more expressive than standard distributed knowledge
- But exponentially more succinct

Generalised distributed knowledge: the extremes

$$\sim_G^{+k} = \bigcup_{H \subseteq G, |H| \geq k} \bigcap_{i \in H} \sim_i$$

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- k = 1: the group considers a state
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$$\sim_G^{+k} = \bigcup_{H \subseteq G, |H| \geq k} \bigcap_{i \in H} \sim_i$$

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$$\sim_G^{+|G|} = \sim_G^D$$

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  - possible if ... ble

standard distributed knowledge

- k = 1: the group considers a state
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Generalised distributed knowledge: the extremes

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  - impossible iff at least one member of the group considers it impossible
  - possible iff [standard distributed knowledge] ble
- k = 1: the group considers a state $\sim_G^{+1} = \sim_G^E$
  - impossible iff all agents in the group considers it impossible
  - possib [general knowledge (everybody knows)] e

Generalised distributed knowledge: conclusions

- Between distributed and general knowledge
  - Intuitively two entirely different concepts
  - But we show that the difference between them can be explained quantitatively rather than qualitatively
  - Specific instances of the same concept, corresponding to which voting threshold is used
  - There is a scale of intermediate concepts between them

Resolving distributed knowledge

Distributed knowledge again

- Common interpretations of distributed knowledge:
  - Knowledge the group could obtain if they had unlimited means of communication
  - "*A group has distributed knowledge of a fact phi if the knowledge of phi is distributed among its members, so that by pooling their knowledge together the members of the group can deduce phi ...*"
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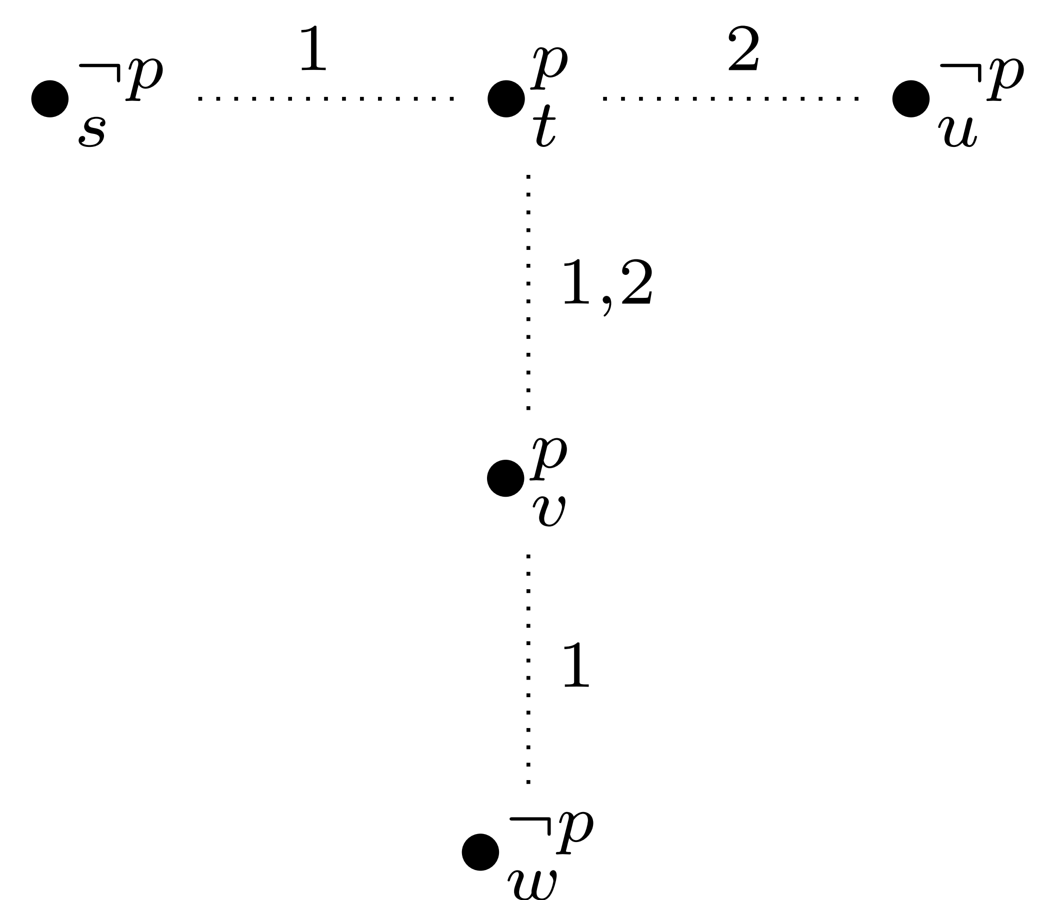

$$D_{1,2}(p \wedge \neg K_1 p)$$

Resolving distributed knowledge (Ågotnes and Wang, 2014)

- Logics with distributed knowledge do not reason about what happens when the group actually share their information
- In this work we introduce a new modality, saying that a formula is true after the group have shared their information - after their distributed knowledge has been resolved

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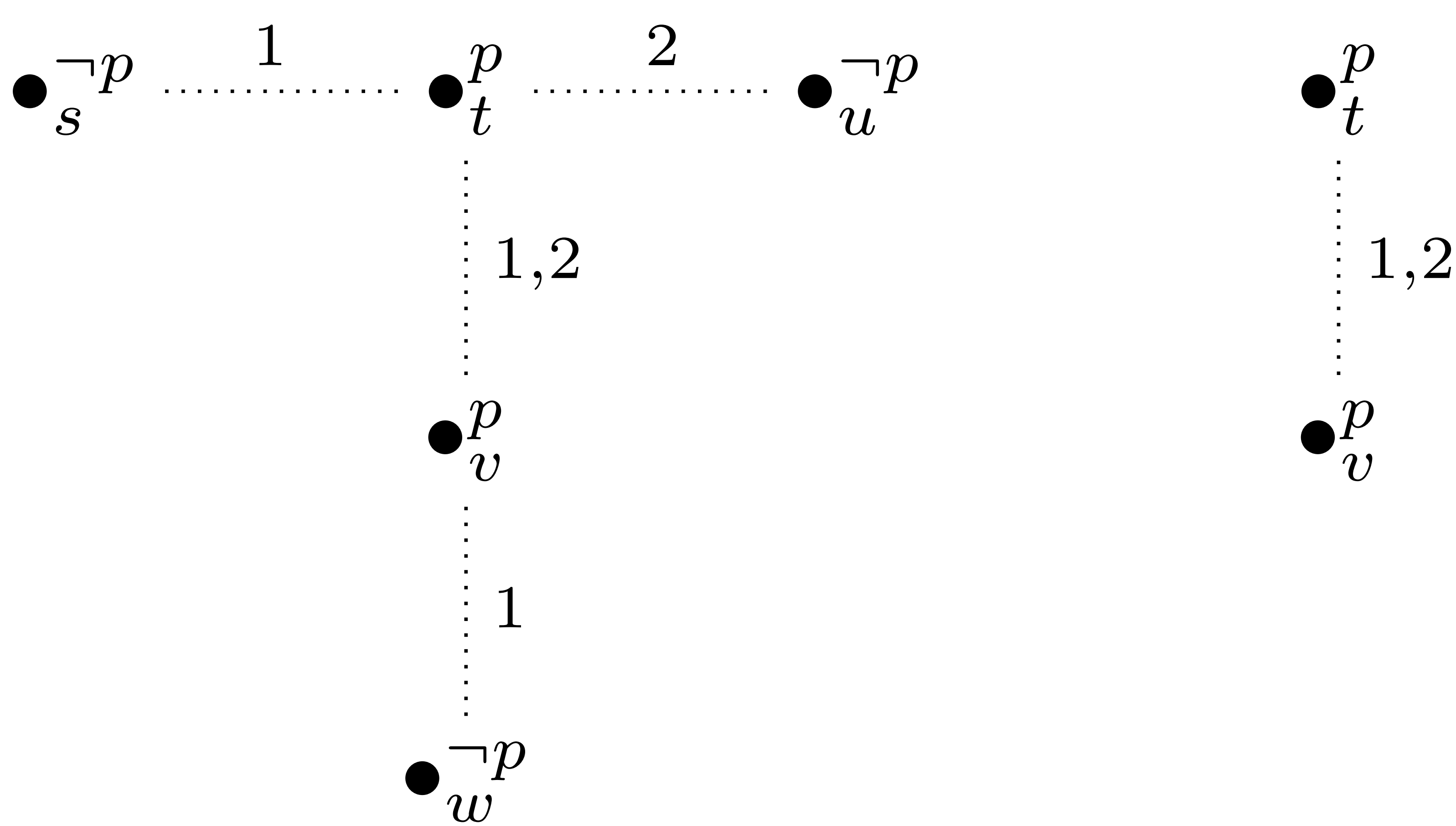

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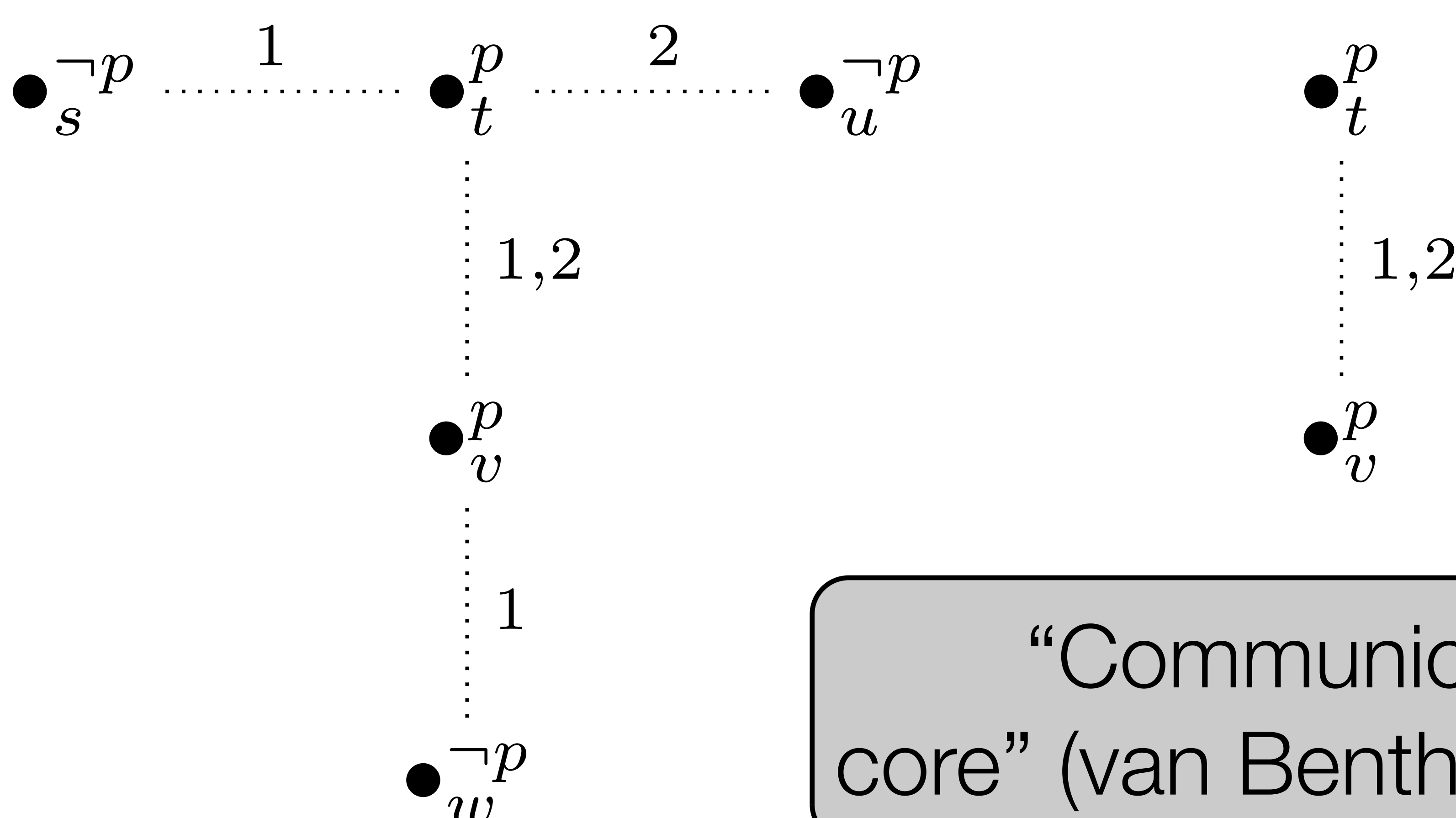

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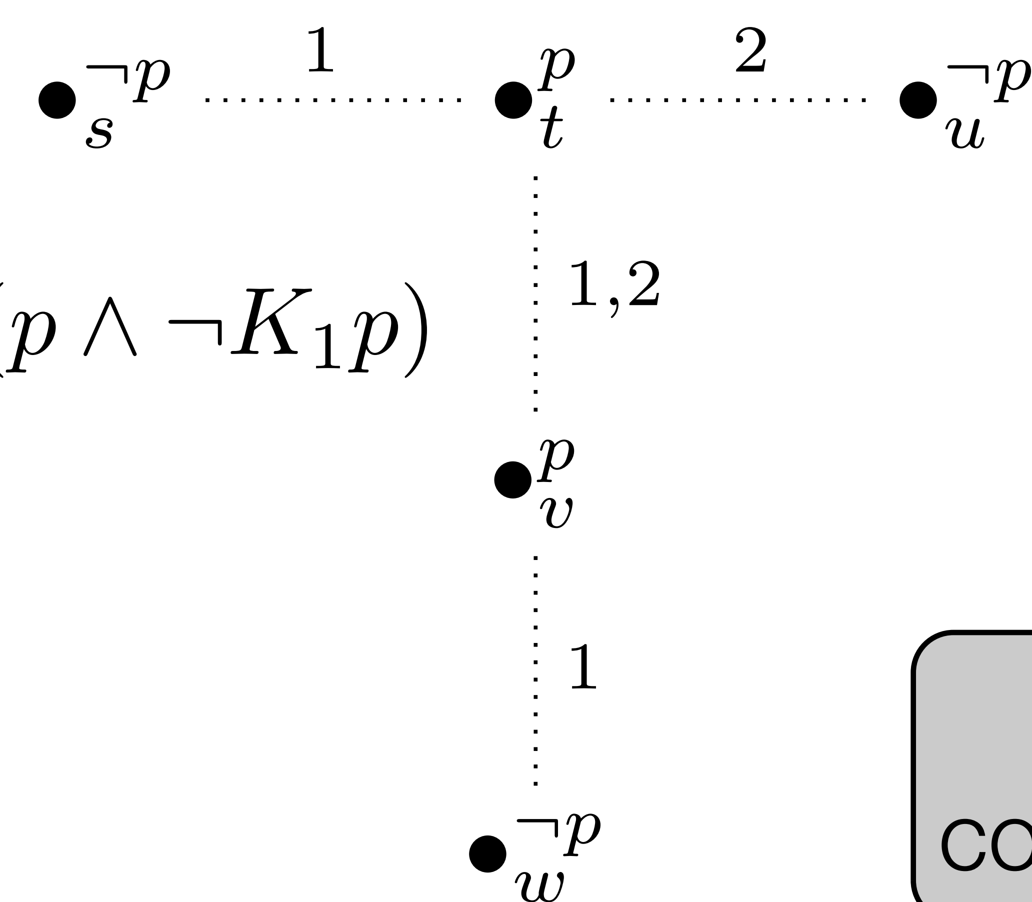

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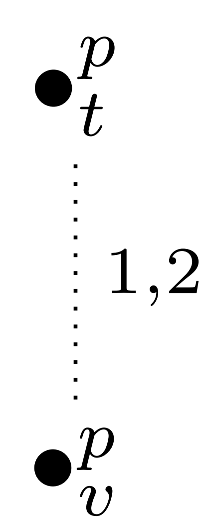

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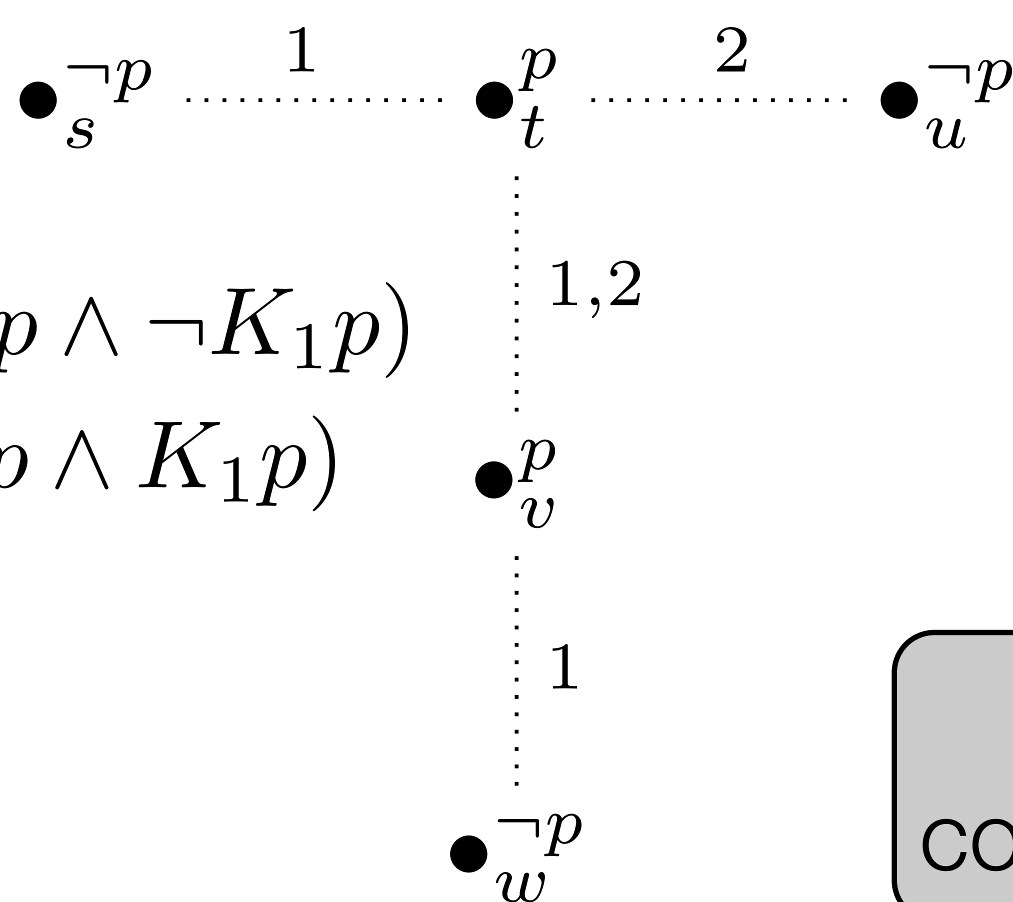
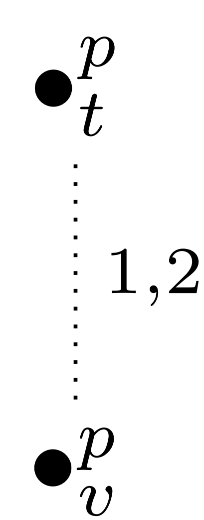


$$M, t \models D_{1,2}(p \land \neg K_1 p)$$
$$M, t \models R_{1,2}(p \land K_1 p)$$



"Communication core" (van Benthem, 2011)

What do other agents know about the fact that a group G resolve their knowledge?

- Will focus here on one (of several) possibilities:
    - It is common knowledge that G resolve their knowledge
        - Semantics: global model update.

Resolving distributed knowledge

$M = (S, \sim_1, \ldots, \sim_n, V)$ (S5 model)

For a group of agents G, the *(global) G-resolved update of* M is the model $M|_G$ where $M|_G = (S', \sim'_1, \ldots, \sim'_n, V')$ and

- $S' = S$
- $\sim'_i = \begin{cases} \bigcap_{j \in G} \sim_j & i \in G \\ \sim_i & \text{otherwise} \end{cases}$
- $V' = V$

Logic

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$$M, s \models R_G\phi \quad \Leftrightarrow \quad M|_G, s \models \phi$$

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$$R_G D_H\phi \leftrightarrow D_H R_G\phi, \text{ when } G \cap H = \emptyset$$

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$$R_G D_H\phi \leftrightarrow D_{G\cup H} R_G\phi, \text{ when } G \cap H \neq \emptyset$$

Axioms (S5)

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$R_G K_i\phi \leftrightarrow D_G R_G\phi$, when $i \in G$

$R_G D_H\phi \leftrightarrow D_H R_G\phi$, when $G \cap H = \emptyset$

$R_G D_H\phi \leftrightarrow D_{G\cup H} R_G\phi$, when $G \cap H \neq \emptyset$

But no similar reduction axiom for $R_G R_H\phi$ (in particular, not equivalent to $R_{G\cup H}\phi$)

Reduction axioms

Proposition: every formula is equivalent to one without resolution operators. The logic is axiomatised by adding the reduction axioms to an axiomatisation of S5 with distributed knowledge.

Adding common knowledge

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid D_G\phi \mid C_G\phi \mid R_G\phi$$

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What about reduction axioms?

For any $G \cap H = \emptyset$, the following is valid:

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What about reduction axioms?

For any $G \cap H = \emptyset$, the following is valid:

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For any $H \subseteq G$ and $i \in G$, the following is valid:

$$R_G C_H \phi \leftrightarrow R_G K_i \phi \leftrightarrow D_G R_G \phi$$

Common knowledge

In general $M, s \models R_G C_H \phi$ iff $M|_G, t \models \phi$ for any $(s,t) \in \sim_H^{*'}$, where

$$\sim_H^{*'} = (\bigcap_{i \in G} \sim_i \cup \bigcup_{i \in H \setminus G} \sim_i)^*$$

– which does not seem to be expressible without the resolution operators

Sound and complete axiomatisation for the case with common and distributed knowledge

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid D_G\phi \mid C_G\phi \mid R_G\phi$$

(K_C)	$C_G(\varphi \to \psi) \to (C_G\varphi \to C_G\psi)$
(T_C)	$C_G\varphi \to \varphi$
(C1)	$C_G\varphi \to E_G C_G\varphi$
(C2)	$C_G(\varphi \to E_G\varphi) \to (\varphi \to C_G\varphi)$
(N_C)	from φ infer $C_G\varphi$.

(S5)	classical proof system for multi-agent epistemic logic
(CK)	axioms and rules for common knowledge
(DK)	characterization axioms for distributed knowledge
(N_R)	from φ infer $R_G\varphi$
(RR)	reduction axioms for resolution (see Proposition 1)
(RR_C)	from $\varphi \to (E_H\varphi \wedge R_{G_0} \cdots R_{G_n} E_H\psi)$ infer $\varphi \to R_{G_0} \cdots R_{G_n} C_H\psi$

Resolution: some open questions

- Expressive power:
    - compare to PACD
    - compare to languages with relativised common knowledge

Public Announcement Logics with Distributed Knowledge

Public Announcement Logic (Plaza, 1989)

$$\varphi ::= p \mid K_i\varphi \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle\varphi_1\rangle\varphi_2$$

ϕ_1 is true, and ϕ_2 is true after ϕ_1 is announced

Formally:

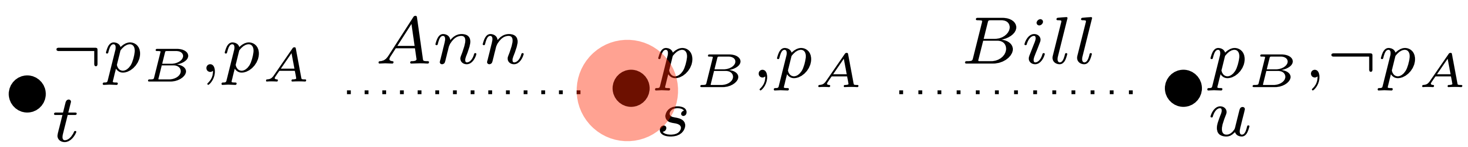
$M = (S, \sim_1, \ldots, \sim_n, V)$ $\quad \sim_i$ equivalence rel. over S

$$M, s \models K_i\phi \quad \Leftrightarrow \quad \forall t \sim_i s\ M, t \models \phi$$
$$M, s \models \langle\phi_1\rangle\phi_2 \quad \Leftrightarrow \quad M, s \models \phi_1 \text{ and } M|\phi_1, s \models \phi_2$$

The model resulting from removing states where ϕ_1 is false

Example

Example



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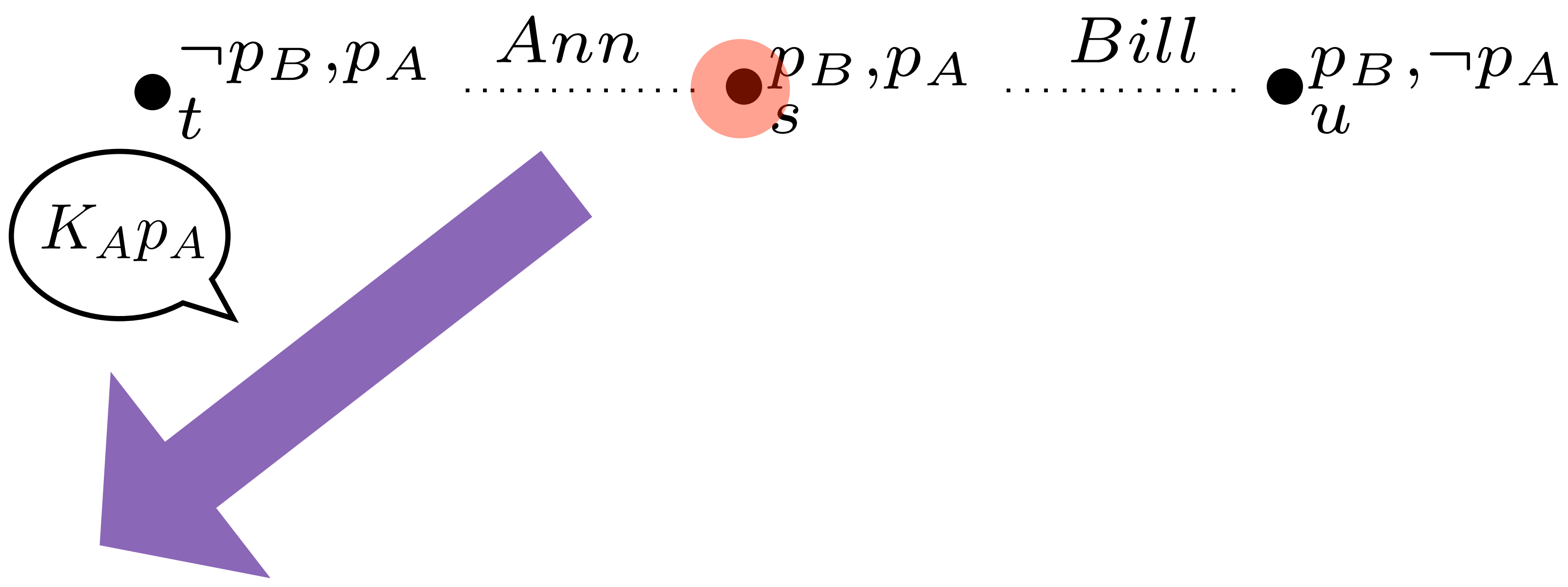

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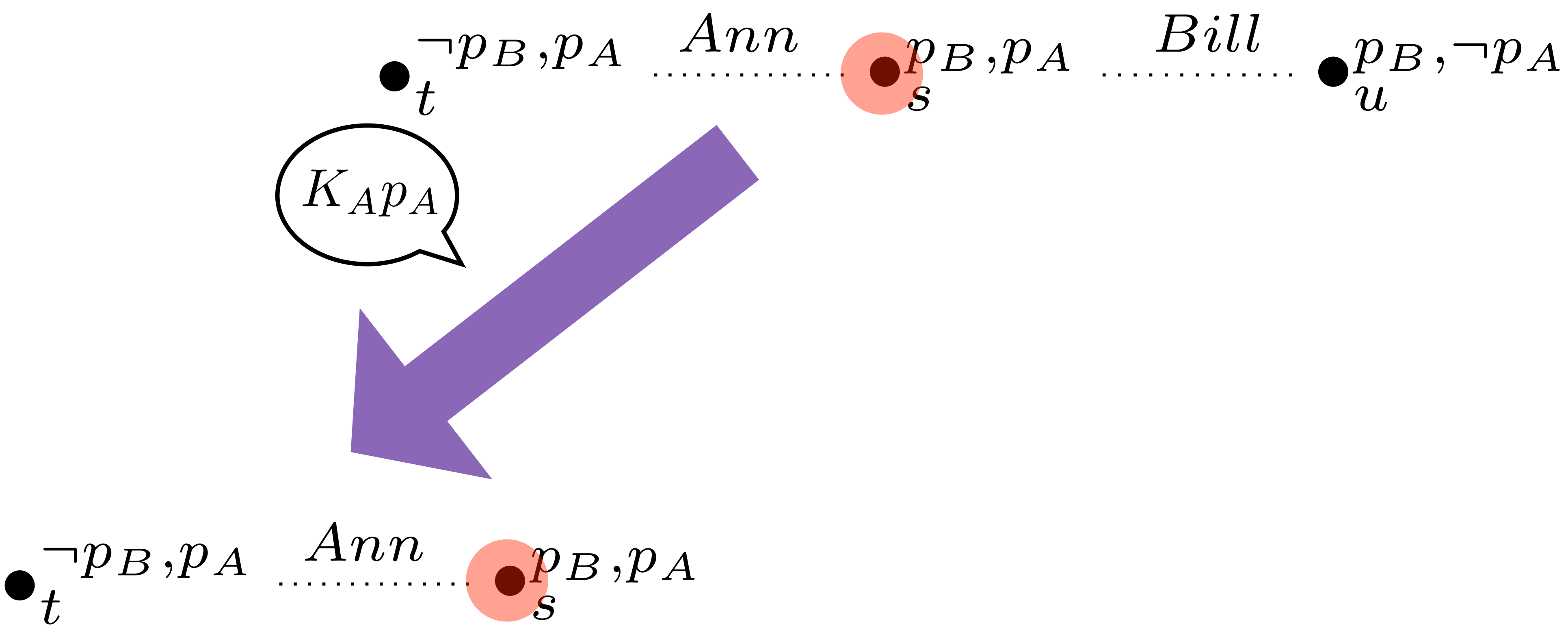

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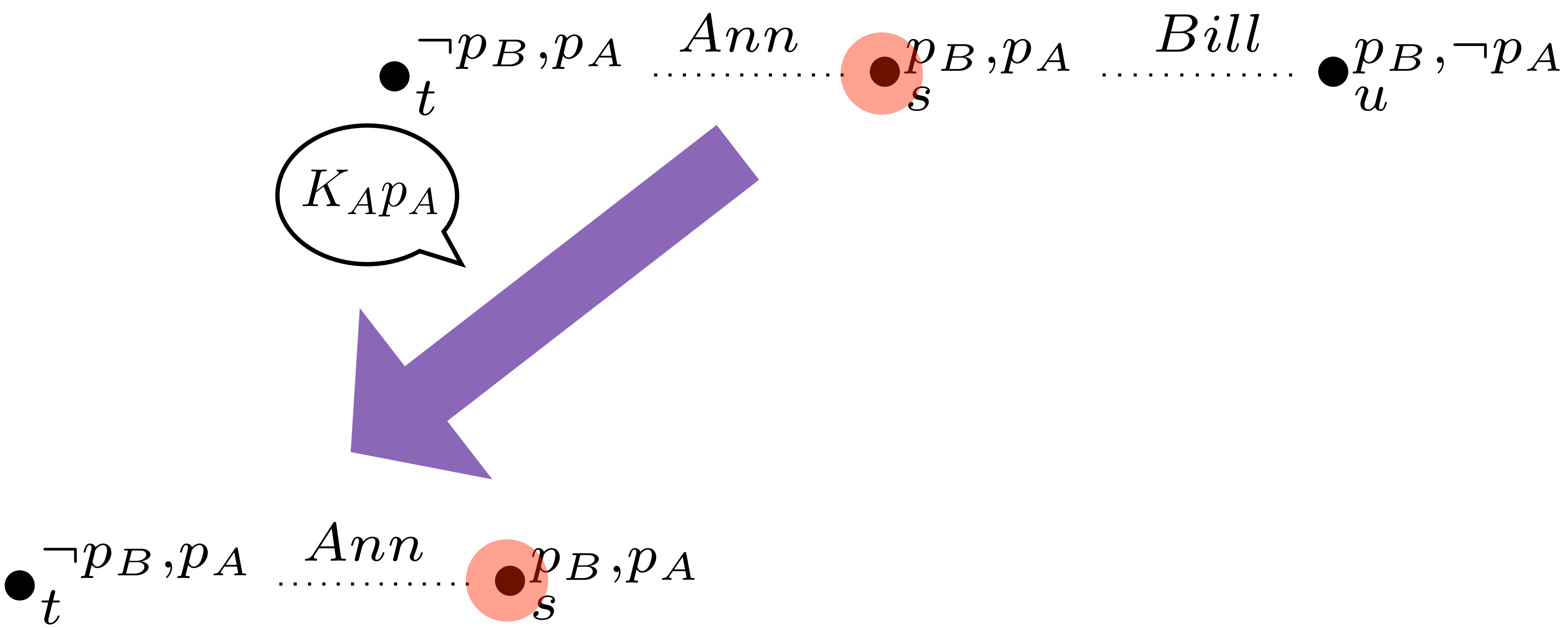
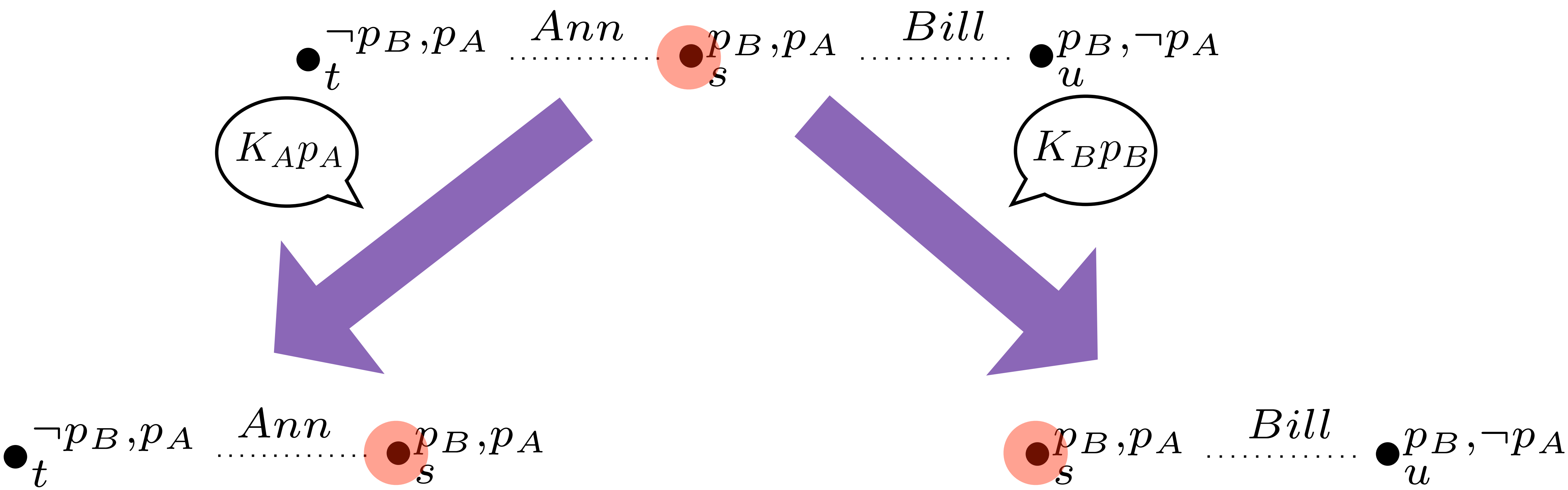


$$M, s \models \langle K_A p_A \rangle K_B p_A$$

Example



$M, s \models \langle K_A p_A \rangle K_B p_A$

$M, s \models \langle K_B p_B \rangle K_A p_B$

Public Announcement Logic with Distributed Knowledge (Wang and Ågotnes, 2013)

- Have not been studied until recently (Wang and Ågotnes, 2013)
- In this work we provide, for different variants of PAL extended with (common and) distributed knowledge
  - Complete axiomatisations
    - No surprises: just add standard axioms
  - Characterisations of expressive power
    - PAD is not more expressive than EL+D
    - PACD is more expressive than both PAC and PAD
  - Characterisations of computational complexity
    - PACD: EXPTIME-complete

$$[\phi]D_A\psi \leftrightarrow (\phi \rightarrow D_A[\phi]\psi)$$

Public Announcement Logic with Distributed Knowledge: completeness proof

- Complications: must deal with, at the same time,
  - S5 knowledge
  - Distributed knowledge (intersection) not modally definable
  - Common knowledge (not canonical)
  - Public announcements
- Develop techniques that might be useful for other purposes (such as resolution operators!)

Group Announcement Logic

Group Announcement Logic (Ågotnes et al., 2010)

Group Announcement Logic extends public announcement logic with:

$$\langle G \rangle \phi : \text{"Group } G \text{ can make an announcement after which } \phi \text{ is true"}$$

Quantification: announcements by an agent

$$K_i\psi$$

Quantification: announcements by an agent

$$M, s \models \langle i \rangle \phi \Leftrightarrow \exists \psi \;\; M, s \models \langle K_i \psi \rangle \phi$$

Quantification: announcements by a group

$$M, s \models \langle G \rangle \phi \quad \Leftrightarrow \quad \exists \{\psi_i : i \in G\}\ M, s \models \langle \textstyle\bigwedge_{i \in G} K_i \psi \rangle \phi$$

Group Announcement Logic (GAL):

$$\varphi ::= p \mid K_i \varphi \mid \neg \varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle \varphi_1 \rangle \varphi_2 \mid \langle G \rangle \phi$$

Example: **The Russian Cards Problem**

From a pack of seven known cards 0,1,2,3,4,5,6 Anne and Bill each draw three cards and Cath gets the remaining card. How can Anne and Bill openly inform each other about their cards, without Cath learning who holds any of their cards?

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Formalisation: 012_a : "Ann has cards 0,1 and 2"

$$(one)\ \bigwedge_{ijk}(ijk_b \to K_a ijk_b) \quad (two)\ \bigwedge_{ijk}(ijk_a \to K_b ijk_a)$$

$$(three)\ \bigwedge_{q=0}^{6}((q_a \to \neg K_c q_a) \wedge (q_b \to \neg K_c q_b))$$

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Known solution:

$anne \equiv 012_a \vee 034_a \vee 056_a \vee 135_a \vee 246_a$

$bill \equiv 345_b \vee 125_b \vee 024_b$

H. van Ditmarsch, *The Russian Cards Problem*, Studia Logica **75**, 2003

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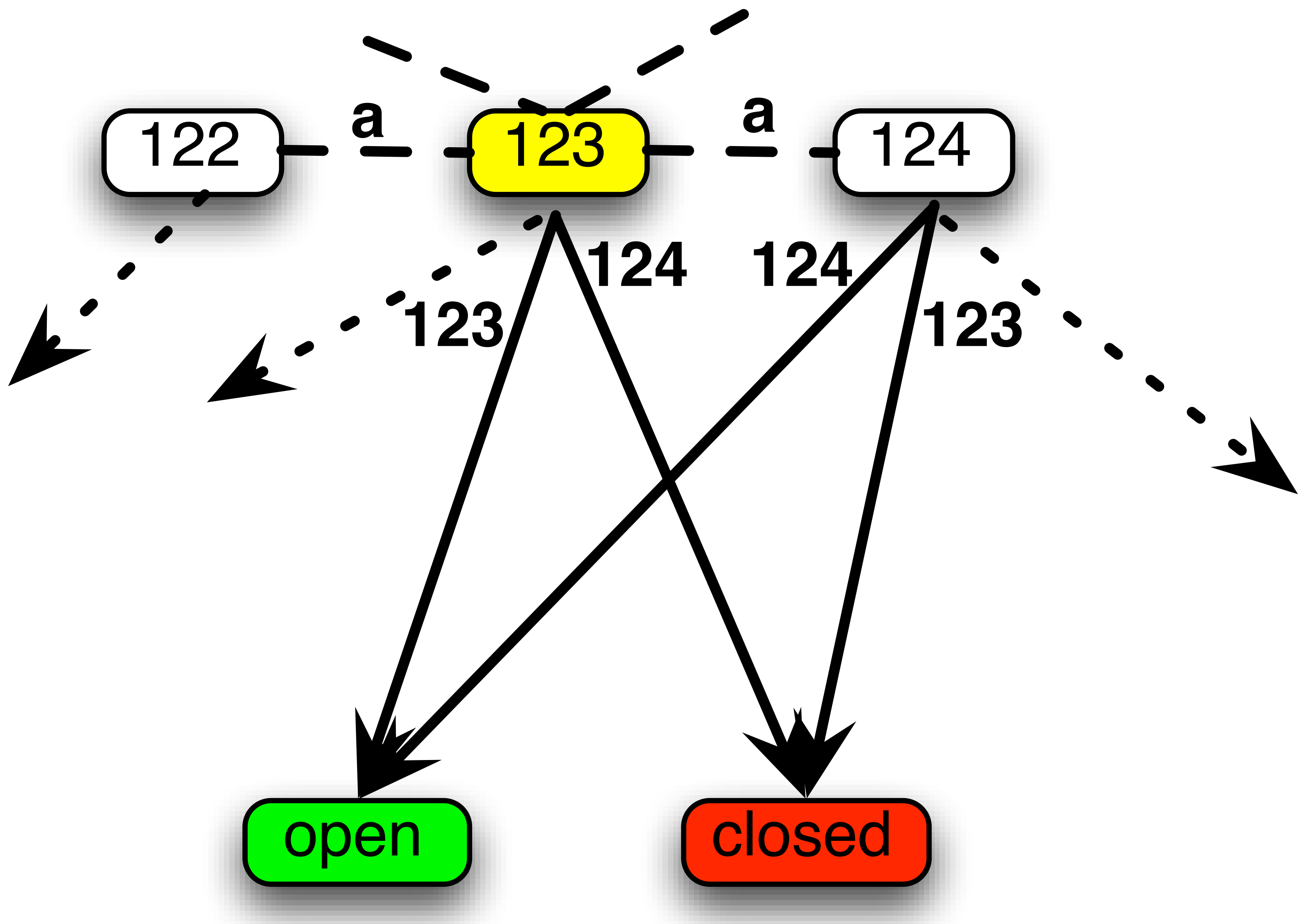
GAL: $\langle a\rangle\langle b\rangle(one \wedge two \wedge three)$

Knowledge and Ability: general actions

- Consider the general case that agents have arbitrary joint actions (and not only group announcements) available, that will take the system to a new state
- Two variants of ability under incomplete information:
  - Knowing *de dicto* that you can achive something: in all the states you consider possible, you can achive the goal (by performing some action)
  - Knowing *de re* that you can achieve something: there is some action which will achieve the goal in all the states you consider possible

Knowledge and Ability: general actions

- Example: agent in front of a combination-lock safe; does not know the combination; correct combination is 123



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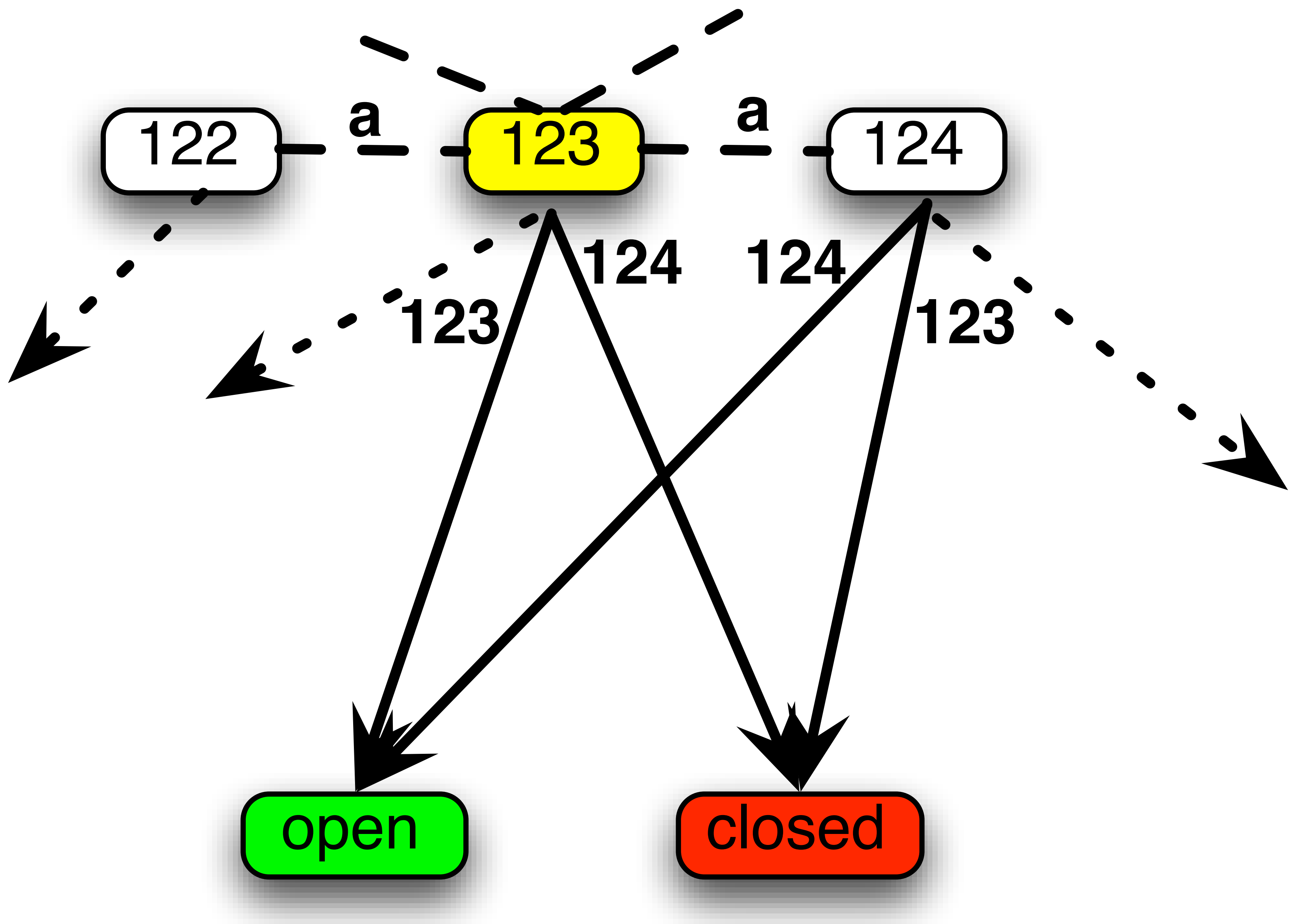

$$\langle a \rangle\, open$$

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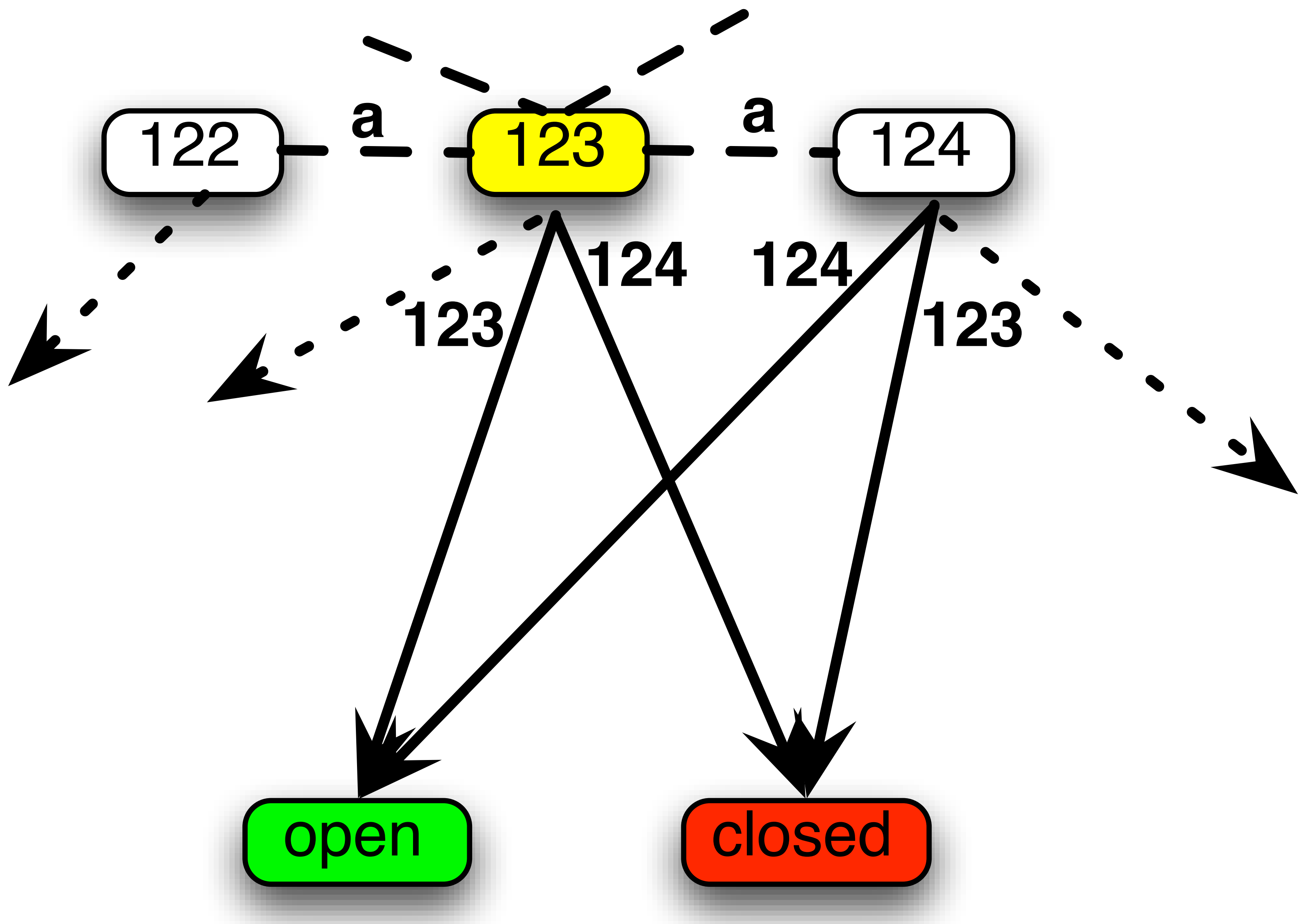

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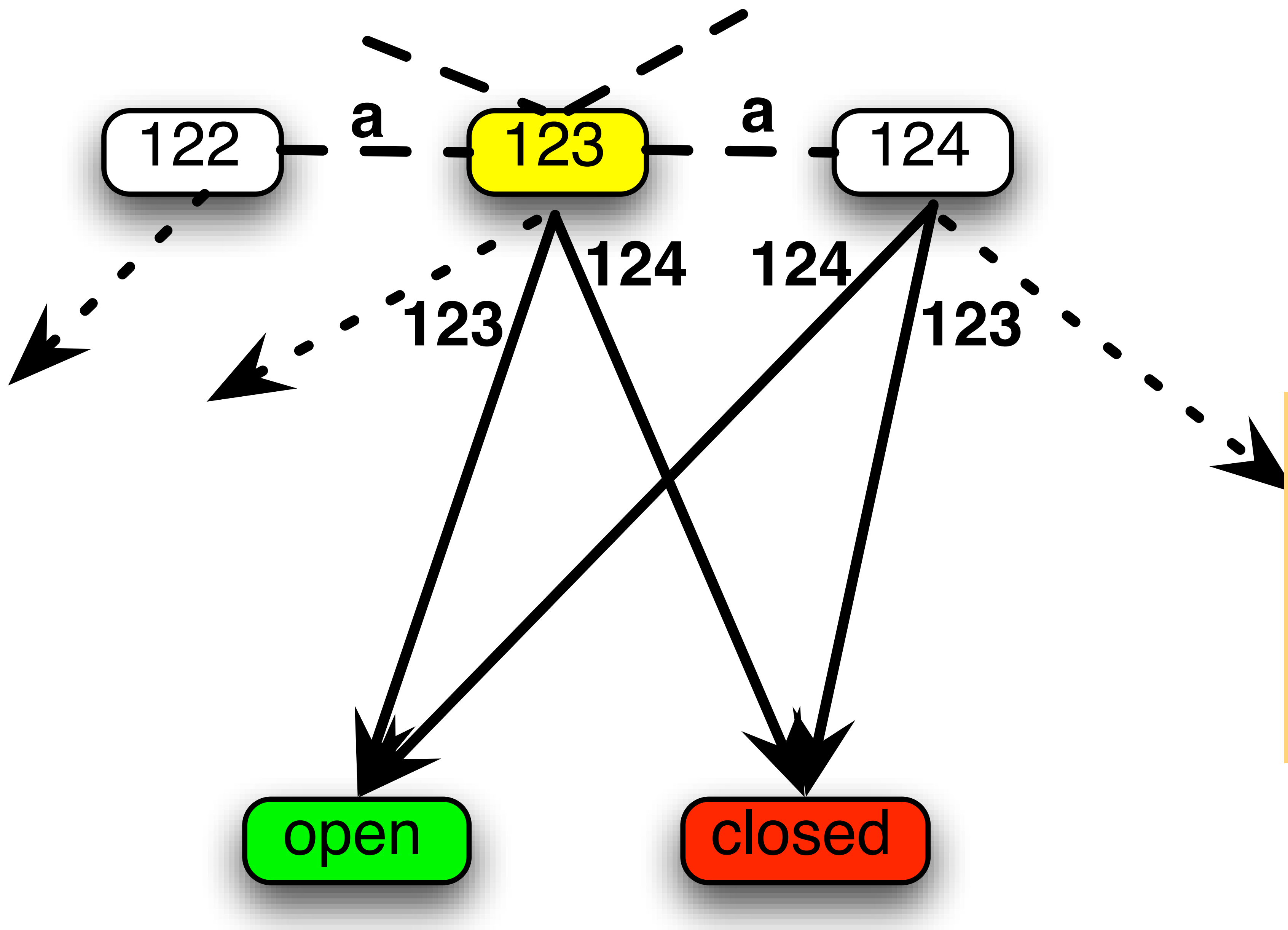

$$\langle a \rangle\, open$$

$$K_a \langle a \rangle\, open$$

But *a* does not know *de re* that she can open the safe

Expressing knowledge *de dicto*/*de re*

Ability	Knowledge of ability, *de dicto*	Knowledge of ability, *de re*
$\exists\psi \; s \models \langle K_a\psi\rangle\phi$	$\forall s \sim_a t \; \exists\psi \; t \models \langle K_a\psi\rangle\phi$	$\exists\psi \; \forall s \sim_a t \; t \models \langle K_a\psi\rangle\phi$
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
$s \models \langle a\rangle\phi$	$s \models K_a\langle a\rangle\phi$	??

Expressing knowledge *de dicto*/*de re*

Ability

$$\exists \psi \ s \models \langle K_a \psi \rangle \phi$$

$$\Updownarrow$$

$$s \models \langle a \rangle \phi$$

Knowledge of ability, *de dicto*

$$\forall s \sim_a t \ \exists \psi \ t \models \langle K_a \psi \rangle \phi$$

$$\Updownarrow$$

$$s \models K_a \langle a \rangle \phi$$

Knowledge of ability, *de re*

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Depends on
(1) the fact that actions are *announcements*
(2) the S5 properties

Group Announcement Logic: some key results

- Complete Hilbert-style axiomatisation (Ågotnes et al., 2010)
- Model checking: PSPACE-complete (Ågotnes et al., 2010)
- Satisfiability/validity: undecidable (co-RE) (Ågotnes, van Ditmarsch and French, 2014)

Sound and Complete Axiomatisation

$S5_n$ axioms and rules

PAL axioms and rules

$[G]\phi \rightarrow [\bigwedge_{i \in G} K_i \psi_i]\phi \qquad$ where $\psi_i \in \mathcal{L}_{el}$

From ϕ, infer $[G]\phi$

From $\phi \rightarrow [\theta][\bigwedge_{i \in G} K_i p_i]\psi$, infer $\phi \rightarrow [\theta][G]\psi$
where $p_i \notin \Theta_\phi \cup \Theta_\theta \cup \Theta_\psi$

Undecidability of GAL: overview

Main steps:

1. enforcing the structure of a satisfying model to have a grid-like structure;
2. defining a formula to represent common knowledge;
3. using propositional atoms to represent tiles, express the formula "it is common knowledge that adjacent tiles on the grid have matching sides".

Undecidability of GAL: overview

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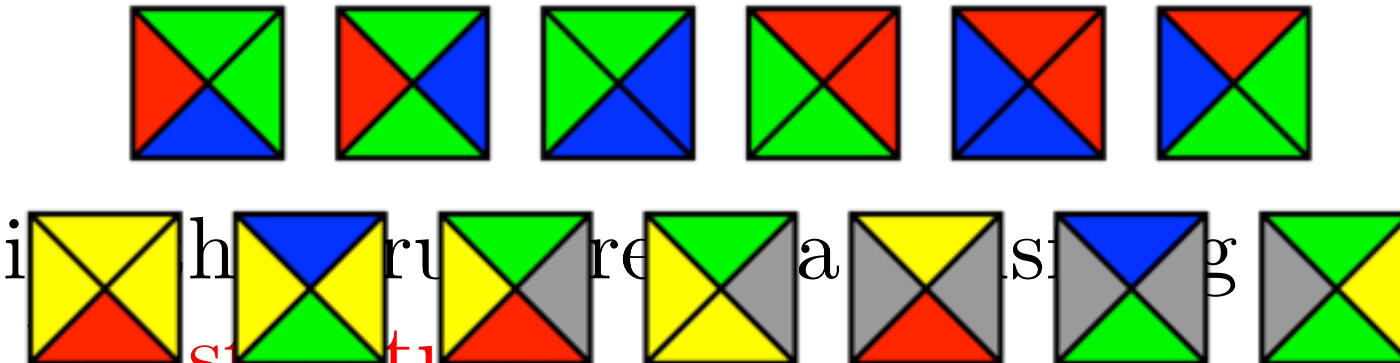

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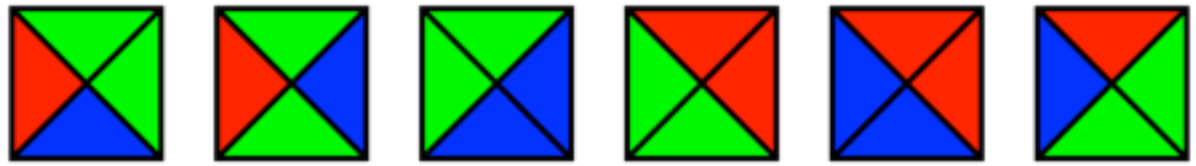
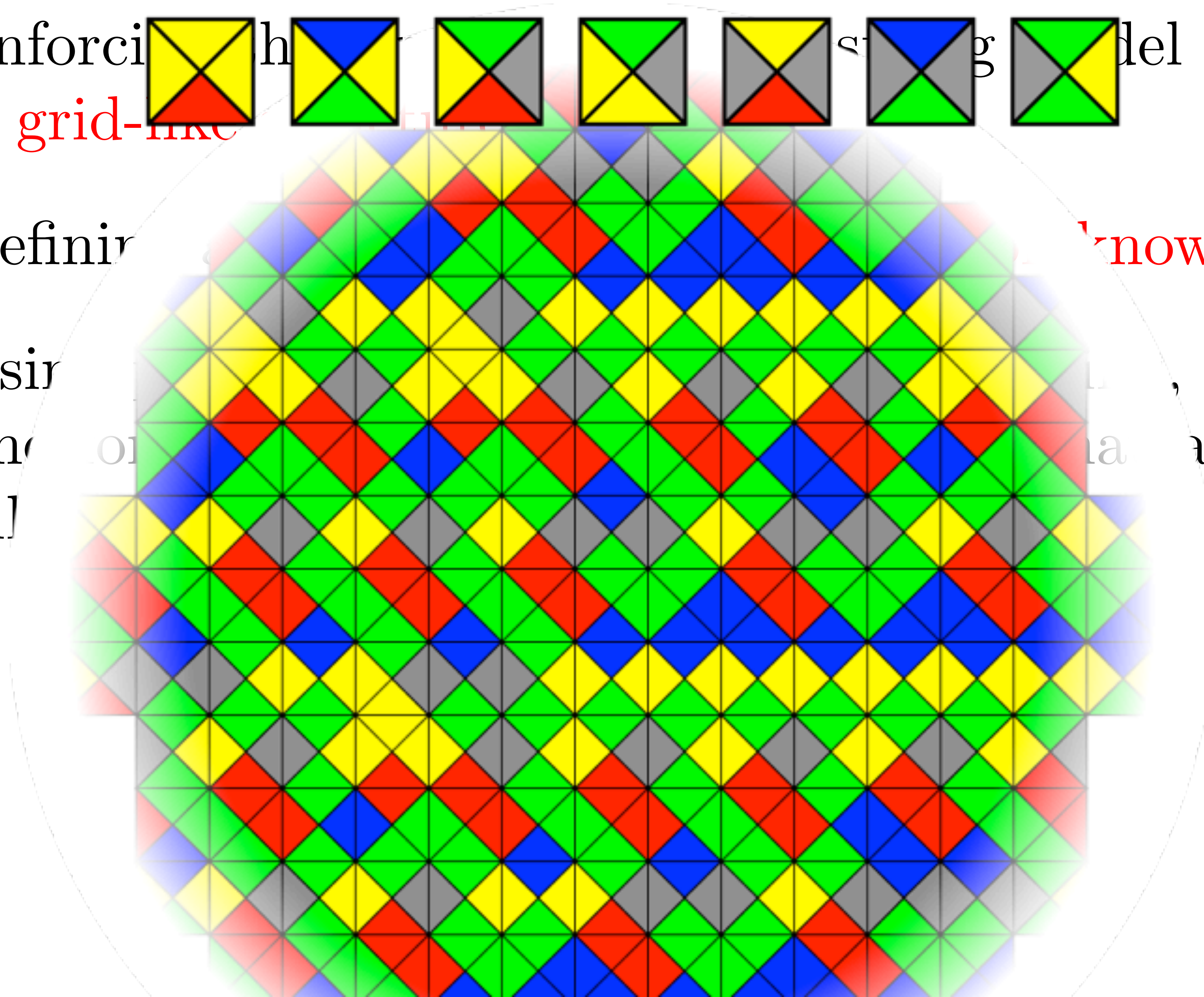


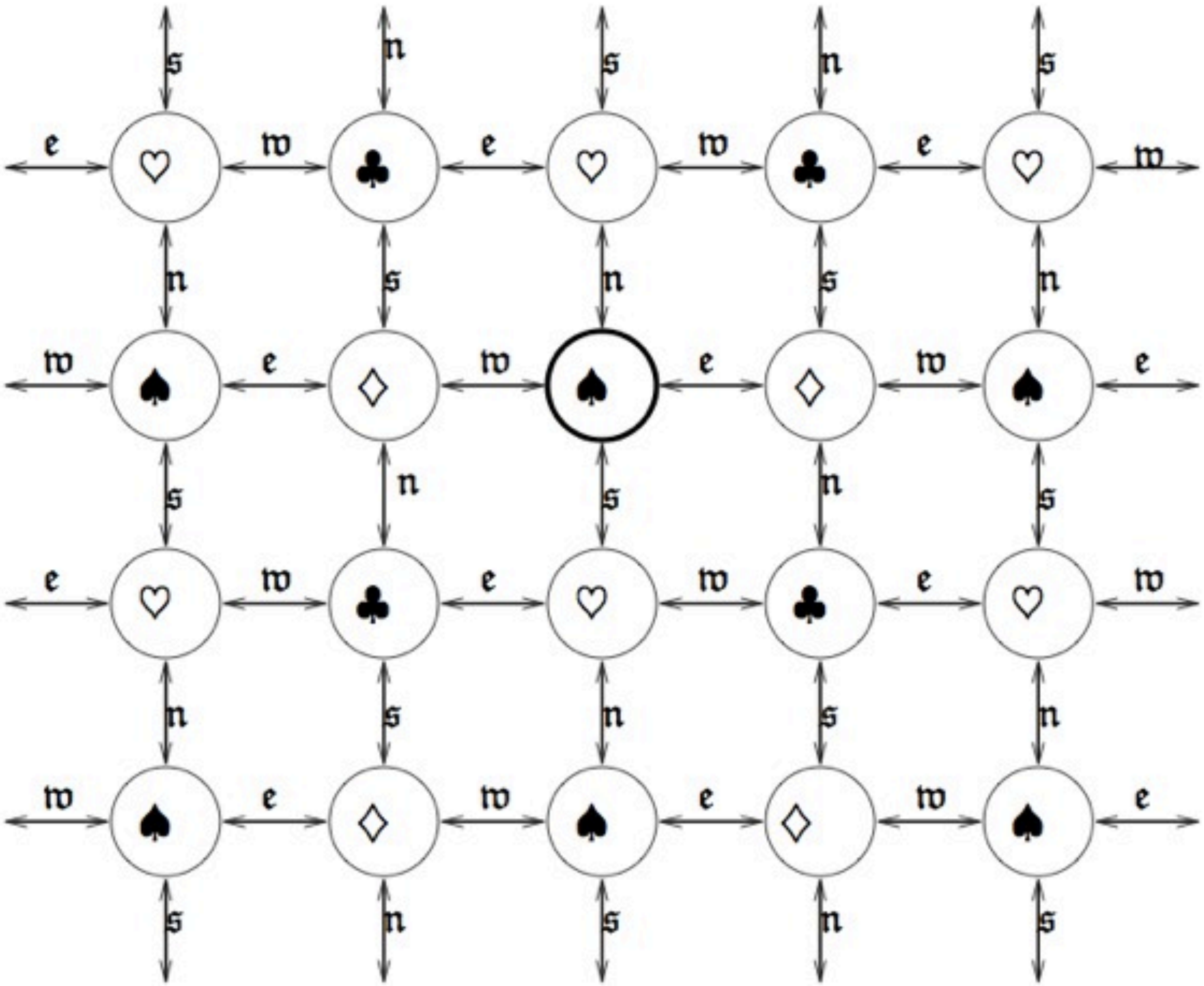
1. enforci... h... s... g ...del to have a grid-like ...



2. definir ... knowledge;

3. usir ... , express the ... o ... a ... adjacent til...

Undecidability of GAL: grid-like structures



Given: a set of tiles Γ

- 5 agents: *East* ($\mathfrak{e}$), *West* ($\mathfrak{w}$), *North* ($\mathfrak{n}$), *South* ($\mathfrak{s}$), and one agent that simulates the common knowledge of the other agents ($\mathfrak{t}$).
- Atomic propositions:
  - $\heartsuit$, $\clubsuit$, $\diamondsuit$and $\spadesuit$
  - p_γ, for each $\gamma \in \Gamma$

GAL: open problems

- (Un)ecidability for less than five agents
- Decidable fragments
- Expressive power compared to Arbitrary Public Announcement Logic (APAL):
    - It is known that GAL is not as expressive as APAL
    - Unknown: can APAL express everything GAL can express (in the multi-agent case)?

Other things: what will they do?

- Which group announcements will rational agents actually make?
- Public announcement games
  - Strategic form (Ågotnes and van Ditmarsch, 2011)
  - Question-answer games (Ågotnes, van Benthem, van Ditmarsch and Minica, 2011)
  - Coalitional (Ågotnes and van Ditmarsch, 2012)

Scientia Potentia Est: on the Power of Knowledge

Consider this scenario:

Consider this scenario:

M: knows that $p \rightarrow q$

T: knows that $r \rightarrow q$

W: knows that $p \wedge r$

p: Robin received the letter

r: the sheriff is at home

q: Robin is by the great oak

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Who knows more?

Who has the most important information?

Who has the most important information about the whereabouts of Robin?

Consider this scenario:

M: knows that $p \rightarrow q$

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W: knows that $p \wedge r$

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Who has the

Who has the most important information about the whereabouts of Robin if communication is possible?

Consider this scenario:

M: knows that $p \to q$

T: knows that $r \to q$

W: knows that $p \wedge r$

p: Robin received the letter

r: the sheriff is at home

q: Robin is by the great oak

the

has the most impo

whereabouts of F

possible?

Will knows more about q, in the sense he can find out q both by talking to Marian and by talking to Tuck

Scientia Potentia Est (Ågotnes, van der Hoek, Wooldridge, 2011)

- Study settings where: information about some objective ("Robin is at the great oak") is distributed among a group of agents, but is typically now known by any individual agent
- We combine:
  - epistemic logic,
  - voting games and power indices
  - to measure how important an agent's information is in an arbitrary subgroup of all agents wrt. the objective
    - Information-based power

Coalitional games

A coalitional game $\Gamma = \langle Ag, \nu \rangle$:

- $Ag = \{1, \ldots, n\}$: set of players
- $\nu : 2^{Ag} \rightarrow \mathbb{R}$ is the characteristic function

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$\nu(C) = 1$: C is winning

Power indices

$$swing(G, i) = \begin{cases} 1 & \text{if } \nu(G) = 0 \text{ and } \nu(G \cup \{i\}) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

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$$\sigma_i = \sum_{G \subseteq Ag \setminus \{i\}} swing(G, i)$$

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Banzhaf measure for agent i:

$$\mu_i = \frac{\sigma_i}{2^{n-1}}$$

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Banzhaf index for agent i:

$$\beta_i = \frac{\sigma_i}{\sum_{j \in Ag} \sigma_j}$$

Power in epistemic models

Given $S = \langle M, s, \chi \rangle$:

- M, s: pointed epistemic model
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power indices

Summing up

- Generalised distributed knowledge
- Resolving distributed knowledge
- Public announcement logic with distributed knowledge
- Group announcement logic
- Scientia Potentia Est

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