A descriptive set-theoretic view of classification problems in operator algebras
- an overview of recent developments

Asger Törnquist

University of Copenhagen
asgert@math.ku.dk

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I. Prologue

Operator algebras 101A, part 1

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Recall that a linear operator $T : H \to H$ is **bounded** if

$$\|T\| = \sup\{\|Tv\| : \|v\| \leq 1\} < \infty,$$

and that T is bounded precisely when it is continuous with respect to the norm on H. The quantity $\|T\|$ is called the **operator norm** of T.

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The set of bounded (linear) operators on H is denoted $\mathcal{B}(H)$.

Operator algebras 101A, part 2

$\mathcal{B}(H)$ is a Banach space (complete normed vector space over $\mathbb{C}$) with the operator norm. Moreover, composition of operators make $\mathcal{B}(H)$ into a *Banach algebra*.

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But $\mathcal{B}(H)$ is a special kind of Banach algebra. The "adjoint" of an operator T, which is the unique operator T^* satisfying

$$\langle Tv, w\rangle = \langle v, T^* w\rangle,$$

is an **involution** on $\mathcal{B}(H)$, which is linear, but anti-multiplicative: $(TS)^* = S^* T^*$.

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- A C^*-**algebra** is a norm-closed $*$-subalgebra of $\mathcal{B}(H)$, for some H;
- Equivalently, a C^*-algebra is a Banach algebra with an involution satisfying the C^*-identity.

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It turns out that it is often fruitful to look not just at $T \in \mathcal{B}(H)$, but at $C^*(T)$, the C^*-algebra **generated** by T.

In fact, $C^*(T)$ is often "too small", containing too few of the operators needed for understanding the structure of T. What we need is a weaker topology than the norm topology.

Operator algebras 101A, part 5

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Operator algebras 101A, part 6

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A **von Neumann algebra** is a weakly closed $*$-subalgebra of $\mathcal{B}(H)$, which includes the identity operator I.

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Clearly $C^*(T) \subseteq W^*(T)$; In most interesting cases, the inclusion is strict.

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A von Neumann algebra $A \subseteq \mathcal{B}(H)$ is a **factor** if the centre of A, i.e.

$$Z(A) = \{T \in A : (\forall S \in A) ST = TS\},$$

consists of scalar multiples of the identity of operator, i.e.,

$$Z(A) = \mathbb{C}I.$$

Operator algebras 101A, part 8

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The moral of this seems to be that:

- Factors are the building blocks of von Neumann algebras.
- Whence our focus should be on *understanding* factors.

Operator algebras 101A, part 9

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Initially, there were type I, II and III, but then over time people refined this to have type I_n, $n \in \{1, 2, 3, \ldots, \infty\}$, type II_1 and type II_∞, and finally, type III_λ, $\lambda \in [0, 1]$.

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But then, over time, more and more infinite families of strange and wonderful factors were found leaving one to wonder: Is it at all possible to classify factors up to isomorphism?

II.
Classification problems from the point of view of Descriptive set theory

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- separable Banach spaces;
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- We give $\mathcal{B}(H)$ the Borel structure generated by the weakly open sets.

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If there is a Borel reduction of E *to* F*, then we say* E *is* **Borel reducible** *to* F*, written* $E \leq_B F$*.*

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- E and F are typically some kind of "**isomorphism relation**" among the objects in X and Y, respectively.
- A Borel reduction $\theta : X \to Y$ of E to F gives a classification of the points of X up to E-equivalence by a Borel assignment of F-classes.

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- The class of Borel functions plays the role of a suitably (very) general class of "calculable" functions.
- If we don't make *any* assumptions on the definability of the reduction θ, then reducibility would just amount to comparing the cardinality of the quotient spaces X/E and Y/F.

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  - We define in Γ the equivalence relation

$$\gamma \simeq^{C^*} \delta \iff C^*(\gamma) \text{ is isomorphic to } C^*(\delta).$$

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Remark: There is another (equivalent) parametrization as a standard Borel space for the separably acting von Neumann algebras, namely the *Effros Borel space*. We will return to this if time allows at the end of the talk.

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Basic fact: The equivalence relations $\simeq^{C^*}$ and $\simeq^{W^*}$ are analytic as subsets of $\Gamma \times \Gamma$. (I.e., there are Borel functions from $\mathbb{N}^{\mathbb{N}}$ onto them.)

N.b.! This fact doesn't rule out that $\simeq^{C^*}$ and $\simeq^{W^*}$ *could* be Borel. It will follow from later results in this talk that they are in fact *not* Borel, but are *complete analytic*.

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  is easily seen to be closed in the product topology (taking $\mathbb{N}$ discrete.)
- **GP** may reasonably be thought of as the **Polish space of countably infinite groups.**

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For $\delta \in S_\infty$, and (f, g, e) we define

$$\delta \cdot f(n, m) = f(\delta^{-1}(n), \delta^{-1}(m)),$$

and

$$\delta \cdot g(n) = g(\delta^{-1}(n)).$$

Then the action

$$\delta \cdot (f, g, e) = (\delta \cdot f, \delta \cdot g, \delta^{-1}(e))$$

is easily seen to induce the isomorphism relation in **GP**.

The Logic Action

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We will say that an action of a Polish group G on a standard Borel space Y is Borel if the map $G \times Y \to Y : (\delta, y) = \delta \cdot y$ is Borel. We will call Y a **Borel G-space**.

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Note: The "logic actions" are continuous actions of S_∞, so they are Borel.

Orbit equivalence relations

Each Borel action $a : G \times Y \to Y$ of a Polish group G on a Polish space Y gives rise to an **orbit equivalence relation** E^a, defined by

$$yE^a y' \iff (\exists g \in G) g \cdot y = y'.$$

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Let F be an equivalence relation on a standard Borel space X. We will say that F is **classifiable by countable structures** *if there is a Borel S_∞-space Y, with a Borel action $a : S_\infty \times Y \to Y$, such that*

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Remark: This definition is motivated by the fact that all S_∞ actions can be described in terms of appropriate "logic actions", for an appropriate choice of structures on $\mathbb{N}$.

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The key notion in this work is the smooth/non-smooth dichotomy, which in our terminology is the following:

Definition

An equivalence relation E on a standard Borel space is called **smooth** if there is a Borel reduction of E to $=_{\mathbb{R}}$, the equality relation in $\mathbb{R}$.

An historical remark, II

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An historical remark, II

The standard example of an equivalence relation which is not smooth is eventual equality on $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$:

$$xE_0y \iff (\exists N)(\forall n \geq N)x_n = y_n.$$

Though E_0 is not smooth, it is hardly a horrible equivalence relation. In fact, being able to classify something by using E_0 classes as invariants would in most fields of mathematics probably be seen as a victory!

An historical remark, III

Borel reducibility is a theory that allows us to go far beyond the smooth/non-smooth dichotomy, and prove that naturally occurring equivalence relations are far, far worse than E_0.

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An historical remark, III

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In fact, in most interesting cases, classification problems turn out to be far worse than E_0. For instance, already isomorphism of countable graphs or groups is far worse than E_0.

Comparing classification problems to isomorphism relations of countable structures is a step in the direction of proving that certain classification problems are not just bad, they are worse.

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- Standard Borel spaces may be used to parametrize all separable C^* and von Neumann algebras acting on a separable complex Hilbert space H.

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- There are also standard Borel spaces of "countable structures", such as groups, graphs, but also countable linear orders, hypergraphs, fields, etc.
- The isomorphism relation in these parametrizations become analytic equivalence relations.
- Borel reducibility gives us a way of comparing equivalence relations on standard Borel spaces, to "*measure their relative complexity*".

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- Can von Neumann algebras be completely classified by assigning countable groups, graphs or other countable structures as invariants?
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- If the answer is no, can we make further determinations of "how bad" classification problems are?

III. Applications to classification problems in operator algebras

Von Neumann algebras

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- II_1 *factors are not classifiable by countable structures.*
- II_∞ *factors are not classifiable by countable structures.*
- *For each* $\lambda \in [0,1]$, *the factors of type* III_λ *are not classifiable by countable structures.*

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Theorem (Sasyk-T., 2009)

ITPFI factors cannot be classified by countable structures.

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Note: Our proof does not seem to give this for type III_λ, but it can be derived for type III_0 by using a recent result of Foreman and Weiss. For type III_λ, $\lambda > 0$ it seems to be open.

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Conjecture (Törnquist): The isomorphism relation for separably acting type II_1 factors is $\leq_B$ universal among orbit equivalence relation induced by the unitary group.

C^*-algebras

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A possible reason is that very complicated invariants are necessary!

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- *The isomorphism relation for countable graphs (and all other types of countable structures) is Borel reducible to isomorphism of amenable, simple, separable, unital C^*-algebras.*
- *In fact, the homeomorphism relation for compact metric spaces is Borel reducible to it.*

What about an upper bound? For the nuclear simple separable unital algebras, an upper bound was provided by an action of the automorphism group of $\mathcal{O}_2$, but the argument was extremely complicated.

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Theorem (Elliott-Farah-Paulson-Rosendal-Toms-T., 2013.)

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Sabok's argument is rather long and complicated (it takes the route of proving that affine homeomorphism of Choquet simplexes is universal, and then employs a theorem by Farah-Toms-T. that says that this equivalence relation is Borel reducible to isomorphism of nuclear, simple, separable C^*-algebras.)

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Theorem (Joseph Zielinski, 2014)

Homeomorphism of compact metric spaces is a universal equivalence relation induced by Polish group actions.

Where to from here?

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- Is there a "turbulence theory" for the unitary group?

The end