Dependence and Independence: a Logical Approach

Applications of team semantics

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Team semantics

Single assignments

$$\Downarrow$$

Sets of assignments

Team semantics

Single assignments

$$\Downarrow$$

Sets of assignments

$$\|$$

Teams

Team semantics

Single assignments

$$\Downarrow$$

Sets of assignments

$$\|$$

Teams

$$\Downarrow$$

Multi-teams

Team semantics

Assignment	
$\begin{array}{c\|c\|c\|c\|} & x & y & z \\ \hline s & 1 & 0 & 2 \end{array}$	

Team semantics

Assignment	
<table><tr><td></td><td>x</td><td>y</td><td>z</td></tr><tr><td>s</td><td>1</td><td>0</td><td>2</td></tr></table>	
<table><tr><td></td><td>**color**</td><td>**shape**</td><td>**height**</td></tr><tr><td>s</td><td>yellow</td><td>wrinkled</td><td>tall</td></tr></table>	

Team semantics

Assignment	Team
$\begin{array}{c\|c\|c\|c\|} & x & y & z \\ \hline s & 1 & 0 & 2 \end{array}$	$\begin{array}{c\|c\|c\|c\|} & x & y & z \\ \hline s_1 & 1 & 0 & 2 \\ s_2 & 2 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ s_n & 1 & 3 & 1 \end{array}$
$\begin{array}{c\|c\|c\|c\|} & \textbf{color} & \textbf{shape} & \textbf{height} \\ \hline s & \text{yellow} & \text{wrinkled} & \text{tall} \end{array}$	

Team semantics

Assignment	Team
(see below)	(see below)

Assignment

	x	y	z
s	1	0	2

Team

	x	y	z
s_1	1	0	2
s_2	2	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
s_n	1	3	1

Assignment

	color	**shape**	**height**
s	yellow	wrinkled	tall

Team

	color	**shape**	**height**
s_1	yellow	wrinkled	tall
s_2	green	wrinkled	short
s_3	green	round	tall

Multi-team semantics

Assignment	Multi-team

Assignment:

	x	y	z
s	1	0	2

Multi-team:

	x	y	z
s_1	1	0	2
s_2	2	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
s_n	1	0	2

Assignment:

	color	**shape**	**height**
s	yellow	wrinkled	tall

Multi-team:

	color	**shape**	**height**
s_1	yellow	wrinkled	tall
s_2	green	wrinkled	short
s_3	green	wrinkled	short

Definition

A *multi-team* is a pair (X, τ), where X is a set and τ is a function such that

1. $\mathrm{Dom}(\tau) = X$,
2. If $i \in X$, then $\tau(i)$ is an assignment for one and the same set of variables. This set of variables is denoted by $\mathrm{Dom}(X)$.

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4. Opens the door to probabilistic teams.

Dependence and independence as atoms

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- Inclusion atom $x \subseteq y$, "values of x occur also as values of y".

Dependence and Independence

Life Sciences	Mendel's Laws, Hardy-Weinberg paradox
Social Sciences	Arrow's theorem
Physical Sciences	Entanglement, non-locality
Computer Science	Database dependence
Mathematics	Linear algebra
Statistics	Random Variables
Logic	Dependence of variables, logical independence
Model theory	Shelah's classification theory

Examples

- I will park the car next to the lamp post depending only on whether it is Thursday or not.
- I will park the car next to the lamp post independently of whether it is past 7 P.M. or not.
- Whether the objects fall to the ground simultaneously depends only on whether they are dropped from the same height or not.
- Whether the objects fall to the ground simultaneously is independent of whether they weigh the same or not.

Examples

- I will park the car next to the lamp post depending only on the day of the week.
- I will park the car next to the lamp post depending only on the day of the week, apart from a few exceptions.
- I will park the car next to the lamp post independently of the day of the week.
- The time of descent of the ball depends only on the height of the drop.
- The time of descent of the ball is independent of the weight of the ball.

Notation

- $x_0, x_1, x_2, \ldots$ individual variables.
- $\mathbf{x}, \mathbf{y}, \ldots$ finite sequences of individual variables.
- $\mathbf{xy}$ means concatenation.

Armstrong's Axioms

1.	Identity rule:	$=\!(x, x)$.
2.	Symmetry Rule:	If $=\!(xt, yr)$, then $=\!(tx, yr)$ and $=\!(xt, ry)$.
3.	Weakening Rule:	If $=\!(x, yr)$, then $=\!(xt, y)$.
4.	Transitivity Rule:	If $=\!(x, y)$ and $=\!(y, r)$, then $=\!(x, r)$.

Axioms of approximate dependence

A1.	$=_0(xy, x)$	(Reflexivity)
A2.	$=_1(x, y)$	(Totality)
A3.	If $=_p(x, yv)$, then $=_p(xu, y)$	(Weakening)
A4.	If $=_p(x, y)$, then $=_p(xu, yu)$	(Augmentation)
A5.	If $=_p(xu, yv)$, then $=_p(ux, yv)$ and $=_p(xu, vy)$	(Permutation)
A6.	If $=_p(x, y)$ and $=_q(y, v)$, where $p + q \leq 1$, then $=_{p+q}(x, v)$	(Transitivity)
A7.	If $=_p(x, y)$ and $p \leq q \leq 1$, then $=_q(x, y)$	(Monotonicity)

Geiger-Paz-Pearl axioms

1.	Empty set rule:	$x \perp \emptyset$.
2.	Symmetry Rule:	If $x \perp y$, then $y \perp x$.
3.	Weakening Rule:	If $x \perp yr$, then $x \perp y$.
4.	Exchange Rule:	If $x \perp y$ and $xy \perp r$, then $x \perp yr$.

Axioms of relative independence

Definition

The **axioms** of the relative independence atom are:

1. $y \perp_x y$ entails $y \perp_x z$ (Constancy Rule)
2. $x \perp_x y$ (Reflexivity Rule)
3. $z \perp_x y$ entails $y \perp_x z$ (Symmetry Rule)
4. $yy' \perp_x zz'$ entails $y \perp_x z$. (Weakening Rule)
5. If z' is a permutation of z, x' is a permutation of x, y' is a permutation of y, then $y \perp_x z$ entails $y' \perp_{x'} z'$. (Permutation Rule)
6. $z \perp_x y$ entails $yx \perp_x zx$ (Fixed Parameter Rule)
7. $x \perp_z y \wedge u \perp_{zx} y$ entails $u \perp_z y$. (First Transitivity Rule)
8. $y \perp_z y \wedge zx \perp_y u$ entails $x \perp_z u$ (Second Transitivity Rule)

Semantics of the dependence atom

Definition

A team X satisfies the atom $=(x, y)$ if

$$\forall s, s' \in X(s(x) = s'(x) \rightarrow s(y) = s'(y)).$$

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$$\forall s, s' \in X(s(x) = s'(x) \to s(y) = s'(y)).$$

Example

$X =$ scientific data about dropping iron balls in Pisa. X satisfies

$$=(\text{height}, \text{time})$$

if in any two drops from the same height the times of descent are the same.

Approximate dependence

Definition

Suppose p is a real number, $0 \leq p \leq 1$. A finite team X is said to satisfy the *approximate dependence atom*

$$=_p(x, y)$$

if there is $Y \subseteq X$, $|Y| \leq p \cdot |X|$, such that the team $X \setminus Y$ satisfies $=(x, y)$. We then write

$$X \models =_p(x, y).$$

Example

- Every finite team satisfies $=_1(x, y)$, because the empty team always satisfies $=(x, y)$.
- $=_0(x, y)$ is equivalent to $=(x, y)$.
- A team of size n always satisfies $=_{1-\frac{1}{n}}(x, y)$.

Semantics of the independence atom

Definition

A team X satisfies the atomic formula $y \perp z$ if for all $s, s' \in X$ there exists $s'' \in X$ such that $s''(y) = s(y)$, and $s''(z) = s'(z)$.

Semantics of the independence atom

Definition

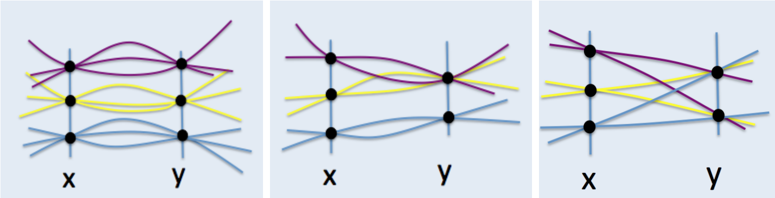
A team X satisfies the atomic formula $y \perp z$ if for all $s, s' \in X$ there exists $s'' \in X$ such that $s''(y) = s(y)$, and $s''(z) = s'(z)$.

Example

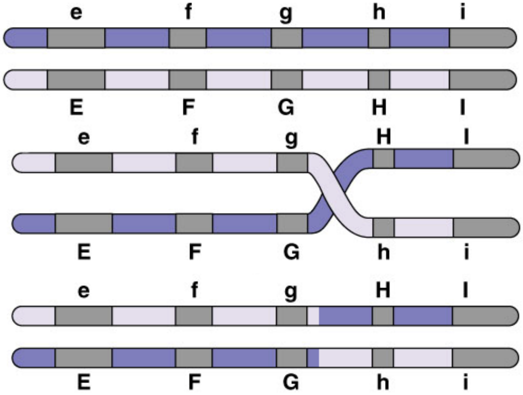
$X =$ scientific experiment concerning dropping iron balls of a fixed size in Pisa. X satisfies

$$\text{weight} \perp \text{height}$$

if for any two drops of a ball also a drop, with weight from the first and height from the second, is performed.



$x = y$ $\qquad$ $=\!(x, y)$ $\qquad$ $x \perp y$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A Completeness Theorem

Theorem (Armstrong)

If T is a finite set of dependence atoms of the form $=\!(u, v)$ for various u and v, then TFAE:

1. $=\!(x, y)$ *follows from T according to the above rules.*
2. *Every team that satisfies T also satisfies $=\!(x, y)$.*

Arguing about pproximate dependence

- The axioms and rules for $=_p(x,y)$ are designed with finite derivations in mind. With infinitely many numbers p we can have infinitary logical consequences (in finite teams), such as
$$\{=_{\frac{1}{n}}(x,y) : n = 1, 2, \ldots\} \models =_0(x,y),$$
which do not follow by the axioms and rules (A1)-(A6)[1].
- We therefore focus on finite derivations and finite sets of approximate dependences.

[1] We can use this example to encode the Halting Problem to the question whether a recursive set of approximate dependence atoms logically implies a given approximate dependence atom.

We have the following Completeness Theorem:

Theorem

Suppose Σ is a finite set of approximate dependence atoms. Then

1. $=_p(x, y)$ *follows from Σ by the above axioms and rules*
2. *Every finite multi-team satisfying Σ also satisfies $=_p(x, y)$.*

A Completeness Theorem

Theorem (Geiger-Paz-Pearl)

If T is a finite set of independence atoms of the form $t \perp r$ for various t and r, then TFAE:

1. $x \perp y$ *follows from* T *according to the above rules*
2. *Every team that satisfies* T *also satisfies* $x \perp y$.

Consequence of relativized independence is undecidable (Herrmann 1995).
Consequence of inclusion is PSPACE-complete (Casanova-Fagin-Papadimitriou 1984).

From database to algebra to model theory

	x	y	z
s_1	1	0	2
s_2	-2	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
s_n	1	3	$\frac{1}{2}$

Logical operations

- Whatever dependence/independence atoms we have, we can coherently add logical operations $\wedge, \vee, \forall$ and $\exists$.
- In front of the atoms can also use $\neg$.
- Conservative extension of classical logic.
- Also: intuitionistic logic, propositional logic, modal logic, etc

Example: disjunction

Definition

A team X satisfies $\phi \vee \psi$ if $X = Y \cup Z$ such that Y satisfies ϕ and Z satisfies ψ.

In strict semantics we require $Y \cap Z = \emptyset$, in lax semantics (default) we do not require this.

Logic of dependence and independence

Definition

1. Dependence logic is the extension of first order logic obtained by adding the dependence atoms $=(x, y)$. (V. 2007)
2. Independence logic is the extension of first order logic obtained by adding the independence atoms $x \perp y$. (Grädel-V. 2010)

Galliani 2012: $=(x, y)$ is definable from $x \perp y$.

- When approximate dependence atoms are added to first order logic we can express propositions such as "the predicate P consists of half of all elements, give or take 5%" or "the predicates P and Q have the same number of elements, with a 1 % margin of error".

The fundamental characterizations

Theorem

- *Dependence logic = existential second order with a* **negative** *predicate for the team. (Kontinen-V. 2009)*
- *Independence logic = existential second order with a predicate for the team. (Galliani 2012)*
- *Finite models: Non-deterministic polynomial time.*

Propositional case

Theorem (Fan Yang 2014)

- *Propositional dependence logic can express all non-void properties of teams that are downward closed.*
- *Propositional dependence logic is equivalent to inquisitive logic of Ciardelli, Groenendijk and Roelofsen.*

New phenomenon: Non-uniform definability

- A connective may be uniformly definable, such as

$$C(\phi, \psi, \theta) \iff (\phi \wedge \psi) \vee (\phi \wedge \theta).$$

- Or just definable, such as

$$X \models \phi \vee_{\mathrm{B}} \psi \iff X \models \phi \text{ or } X \models \psi.$$

- Namely, every instance of $\vee_{\mathrm{B}}$ is individually definable, but $\vee_{\mathrm{B}}$ is not uniformly definable. (F. Yang 2014)
- Truth functional completeness has a new dimension: Every downward closed set of teams is definable but some natural operations on such sets are not definable.

Inclusion atom

- $x \subseteq y$ "values of x are also values of y"
- A directed graph contains a cycle (or an infinite path) iff it satisfies $\exists x \exists y (y \subseteq x \wedge xEy)$

Theorem (Galliani-Hella 2013)

Inclusion logic = Fixpoint logic on finite models
Inclusion logic = PTIME on finite ordered models.

Theorem (Hannula-Kontinen 2014)

Inclusion logic with strict semantics = NP on finite models.

Examples

Example

- $\forall x \forall y \exists z (=\!(z, y) \wedge \neg z = x)$ characterizes infinity.
- Alternatively: $\forall z \forall x \exists y \forall u \exists v (xy \perp uv \wedge (x = u \leftrightarrow y = v) \wedge \neg v = z)$.

Examples

Example

- $\forall x \forall y \exists z(=\!(z, y) \wedge \neg z = x)$ characterizes infinity.
- Alternatively: $\forall z \forall x \exists y \forall u \exists v(xy \perp uv \wedge (x = u \leftrightarrow y = v) \wedge \neg v = z)$.
- $\exists x \exists y(y \subseteq x \wedge y < x)$ characterizes non-well-foundedness.

- Cannot axiomatize logical consequence.
- **Can** axiomatize first order consequences.

The rules

Definition

- Natural deduction of classical logic, but *Disjunction Elimination Rule* and *Negation Introduction Rule* only for first order formulas.
- Weak Disjunction Rule: From $\psi \vdash \theta$ conclude $\phi \vee \psi \vdash \phi \vee \theta$.
- Dependence Introduction Rule: $\exists y \forall x \phi(x, y, \vec{z}) \vdash \forall x \exists y (=\!(\vec{z}, y) \wedge \phi(x, y, \vec{z}))$.
- Dependence Distribution rule
- Dependence Elimination Rule

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Theorem (Completeness Theorem)

The above axioms and rules are complete with respect to the team semantics. (Kontinen-V. 2011)

Social choice

- A field where dependence and independence concepts arise naturally is the theory of social choice.
- Suppose we have n voters $x_1, \ldots, x_n$, each giving his or her (linear) preference quasi-order $<_{x_i}$ on some finite set A of alternatives. We call such sequences $p_1, \ldots, p_n$ **profiles**.

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- Let us denote the social well-fare function by y, which is likewise a preference order $<_y$.
- Naturally we assume

$$=\!(x_1, \ldots, x_n, y).$$

1. A team is **Paretian** if the team satisfies the first order formula:

$$(a <_{x_1} b \wedge \ldots \wedge a <_{x_n} b) \to a <_y b,$$

   for all $a, b \in A$. Note that this means that every individual row satisfies the formula.

2. A team is **dictatorial** if in the team

$$x_1 = y \vee_B \ldots \vee_B x_n = y.$$

3. A team **respects independence of irrelevant alternatives** if it satisfies for all $a, b \in A$:

$$=\!(\{a <_{x_1} b, \ldots, a <_{x_n} b\}, \{a <_y b\}).$$

   Note that this is a Boolean dependence atom.

4. A team supports **voting independence**, if it satisfies for all i:

$$x_i \perp \{x_j : j \neq i\}.$$

Definition

We introduce a new *universality atom* $\forall(x_1, \ldots, x_n)$ with the intuitive meaning that any combination of values (in the given domain) for $x_1, \ldots, x_n$ is possible. A team X satisfies

$$\forall(x_1, \ldots, x_n),$$

if for every $a_1, \ldots, a_n \in M$ there is $s \in X$ such that $s(x_1) = a_1, \ldots, s(x_n) = a_n$.

Axioms for the universality atoms are:

1. $\forall(xy)$ implies $\forall(x)$ (Weakening)
2. $\forall(xy)$ implies $\forall(yx)$ (Symmetry)

Approximate universality: All values occur, apart from p-few exceptions.

Lemma

Suppose $\mathfrak{M} \models_X \forall(x_1) \wedge \ldots \wedge \forall(x_n) \wedge \bigwedge_{i=1}^{n}(x_i \perp \{x_j : j \neq i\})$. *Then* $\mathfrak{M} \models_X \forall(x_1, \ldots, x_n)$.

Could be taken as an axiom.

Proof.

Let $a_1, \ldots, a_n \in M$ be given. Because

$$\mathfrak{M} \models_X \bigwedge_{i=1}^{n} \forall(x_i),$$

there are $s_i \in X$ such that $s_i(x_i) = a_i$ for all $1 \leq i \leq n$. Using

$$\mathfrak{M} \models_X \bigwedge_{i=1}^{n} x_i \perp \{x_j : j \neq i\}$$

we can construct inductively $s'_1, \ldots, s'_n \in X$ such that

1. $s'_1 = s_1$,
2. $s'_{i+1}(x_{i+1}) = s_{i+1}(x_{i+1})$,
3. $s'_{i+1}(x_j) = s'_i(x_j)$, for $j \neq i + 1$.

It follows that $s'_n(x_1) = a_1, \ldots, s'_n(x_n) = a_n$. □

In the social choice context

$$\forall(x_1) \wedge \ldots \wedge \forall(x_n) \wedge \bigwedge_{i=1}^{n}(x_i \perp \{x_j : j \neq i\})$$

says: "We consider the possibility that for any voter and any preference order there is some profile (voting result, row) in which that voter voted that preference order, but we assume that the voters choose their preference orders independently of each other", which seems reasonable. Let us call the assumption

$$\forall(x_1) \wedge \ldots \wedge \forall(x_n)$$

the **freedom of choice** assumption. Together with voting independence it implies, by the previous Lemma, that all patterns of voting can arise.

Theorem (Arrow 1963)

Voting independence, freedom of choice, Pareto and respect of independence of irrelevant alternatives together imply dictatorship. In symbols,

$$
\begin{array}{l}
\{=\!(x_1,\ldots,x_n,y),\\
\bigwedge_{a,b\in A}((a\le_{x_1} b\wedge\ldots\wedge a\le_{x_n} b)\to a\le_y b),\\
\bigwedge_{a,b\in A}=\!(\{a\le_{x_1} b,\ldots,a\le_{x_n} b\},\{a\le_y b\}),\\
\forall(x_1),\ldots,\forall(x_n),\bigwedge_{i=1}^n x_i\perp\{x_j:j\ne i\}\}\\
\qquad\qquad\qquad\qquad\models x_1=y\vee_B\ldots\vee_B x_n=y.
\end{array}
$$

Physics, joint work with Abramsky

- Quantum physics provides a rich field of highly non-trivial dependence and independence concepts. Some of the most fundamental questions of quantum physics are about dependence and independence of outcomes of experiments.
- Bell inequalities imply that the correlation which is observed between the measurements of the spin of two entangled particles along different axis cannot be realized by a function that assigns to any direction in space a definite value which is the value of the spin along the given direction.

Observational teams

One of the intuitions behind the concept of a team is a set of observations, such as readings of physical measurements. Let us consider a experiments

$$q_1, \ldots, q_n.$$

Each experiment has an input x_i and an output y_i.

Observational teams

After m rounds of making the experiments $q_1,\ldots,q_n$ we have the data

$$X = \begin{array}{|ccccc|} x_1 & y_1 & \ldots & x_n & y_n \\ \hline a_1^1 & b_1^1 & \ldots & a_n^1 & b_n^1 \\ a_1^2 & b_1^2 & \ldots & a_n^2 & b_n^2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_1^m & b_1^m & \ldots & a_n^m & b_n^m \end{array}$$

Determinism

Using the dependence atom $=\!(\vec{x}, \vec{y})$ we can say that the team of data X supports **strong determinism** if it satisfies

$$=\!(x_i, y_i)$$

for all $i = 1, ..., n$.
Respectively, we can say that the team X supports **weak determinism** if it satisfies

$$=\!(x_1, ..., x_n, y_i)$$

for all $i = 1, ..., n$.

Hidden variables

An important role in models of quantum physics is played by the so-called **hidden variables**, variables that have an unobservable outcome and no input. In the presence of a hidden variable z we redefine strong determinism as $=\!(\vec{x}z, \vec{y})$, rather than just $=\!(\vec{x}, \vec{y})$, and weak determinism as $=\!(x_i z, y_i)$, rather than just $=\!(x_i, y_i)$.

A **hidden variable team** is a team of the form

$$Y = \begin{array}{|cccccccc|}
x_1 & y_1 & \ldots & x_n & y_n & z_1 & \ldots & z_l \\
\hline
a_1^1 & b_1^1 & \ldots & a_n^1 & b_n^1 & \gamma_1^1 & \ldots & \gamma_l^1 \\
a_1^2 & b_1^2 & \ldots & a_n^2 & b_n^2 & \gamma_1^2 & \ldots & \gamma_l^2 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_1^m & b_1^m & \ldots & a_n^m & b_n^m & \gamma_1^m & \ldots & \gamma_l^m
\end{array}$$

where γ_j^i are the hidden variables.

Single-valuedness

A team X is said to support **single-valuedness of the hidden variable** z if z has only one value in the team.

We can express this with the formula

$$=\!(z).$$

A team X is said to support **no-signalling** if the following holds: Suppose the team X has two measurement-outcome combinations s and s' with input x_i the same. So now $s(y_i)$ is a possible outcome of experiment q_i in view of X. We demand that $s(y_i)$ is also a possible outcome if the inputs $s(x_j)$, $j \neq i$, are changed to $s'(x_j)$.
We can express no-signalling with the formula

$$y_i \perp_{x_i} \{x_j : j \neq i\}.$$

An empirical team X is said to support **z-independence** if the following holds: Suppose the team X has two measurement-outcome combinations s and s'. Now the hidden variable z_i has some value $s(z_i)$ in the combination s. We demand that $s(z_i)$ should occur as the value of the hidden variable also if the inputs $s(\vec{x})$ are changed to $s'(\vec{x})$.
We can express z-independence with the formula

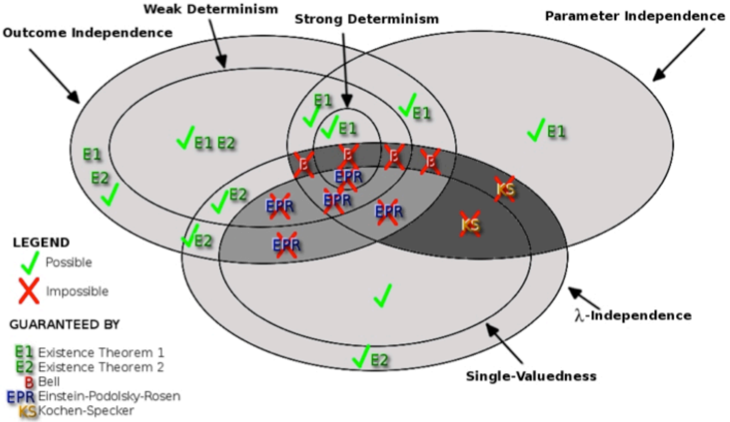
$$z_i \perp \vec{x}.$$

An empirical team X is said to support **Outcome-independence** if the following holds: Suppose the team X has two measurement-outcome combinations s and s' with the same total input data $\vec{x}$ and the same hidden variable z_k, i.e. $s(\vec{x}) = s'(\vec{x})$ and $s(z) = s'(z)$. We demand that output $s(y_i)$ should occur as an output also if the outputs $s(\{y_j : j \neq i\})$ are changed to $s'(\{y_j : j \neq i\})$.
We can express output-independence with the formula

$$y_i \perp_{\vec{x}z} \{y_j : j \neq i\}.$$

An empirical team X is said to support **parameter-independence** if the following holds: Suppose the team X has two measurement-outcome combinations s and s' with the same input data about x and the same hidden variable z_k, i.e. $s(x) = s'(x)$ and $s(z) = s'(z)$. We demand that output $s(y_i)$ should occur as a possible output also if the inputs $s(\{x_j : j \neq i\})$ are changed to $s'(\{x_j : j \neq i\})$.
We can express parameter-independence with the formula

$$\{x_j : j \neq i\} \perp_{x_i z} y_i$$



Picture by Noson Yanofsky in "A Classification of Hidden-Variable Properties", Workshop on Quantum Logic Inspired by Quantum Computation, Indiana, 2009.

Lemma

Weak determinism implies outcome independence.

Proof.

Weak determinism is $=\!(\vec{x}z, y)$ and outcome independence is $y_i \perp_{\vec{x}z} \{y_j : j \neq i\}$.
The Constancy Rule says: $=\!(\vec{x}, y) \models y \perp_{\vec{x}} z$.
By substituting $\vec{x}z$ to $\vec{x}$ we get:

$$=\!(\vec{x}z, y) \models y_i \perp_{\vec{x}z} \{y_j : j \neq i\},$$

as desired. □

Lemma

Strong determinism implies parameter independence.

Proof.

This is again just the Constancy Rule.

$$=\!(x_i, y_i) \models \{x_j : j \neq i\} \perp_{x_i z} y_i.$$

$\square$

Lemma

Parameter independence and weak determinacy imply strong determinacy.

Proof.

We want

$$=(\vec{x}z, y_i) \wedge \{x_j : j \neq i\} \perp_{x_i z} y_i \models =(x_i z, y_i)$$

This is an instance of the First Transitivity Rule

$$y \perp_{\vec{u}z} \vec{w} \wedge \vec{u} \perp_z y \models y \perp_z \vec{w},$$

where $\vec{u} = \{x_j : j \neq i\}$, $y = y_i$ and $z = x_i z$. □

No-go results

No-go results are constructions of special teams.
A hidden variable team Y **realizes** the team X if

$$s \in X \iff \exists s' \in Y(s'(x_1) = s(x_1) \wedge s'(y_1) = s(y_1) \wedge \ldots$$

$$s'(x_n) = s(x_n) \wedge s'(y_n) = s(y_n)).$$

- Einstein-Podolsky-Rosen paradox: There is an empirical model (team) which cannot be realized by any hidden variable model satisfying single-valuedness of the hidden variable and outcome-independence.

Proof.

We consider a system in which the input x_1 is constant 0 and the input x_2 is constant 1. The output in both can be a or b. Let

$$X = \begin{array}{|cccc|} x_1 & y_1 & x_2 & y_2 \\ \hline 0 & a & 1 & b \\ 0 & b & 1 & a \end{array}$$

Suppose this is realized by a hidden variable model

$$X = \begin{array}{|ccccc|} x_1 & y_1 & x_2 & y_2 & z \\ \hline 0 & a & 1 & b & \lambda_1 \\ 0 & b & 1 & a & \lambda_2 \end{array}$$

Single-valuedness implies $\lambda_1 = \lambda_2$. Output-independence fails because the row

$$\begin{array}{|ccccc|} x_1 & y_1 & x_2 & y_2 & z \\ \hline 0 & a & 1 & a & \lambda_1 \end{array}$$

is missing. □

Bell's Theorem 1964

- Bell's Theorem in quantum foundations and quantum information theory, the basis of quantum computation, can be seen as the existence of a team, even arising from real physical experiments, violating a dependence logic sentence, which expresses the (falsely) assumed locality of quantum world. (Joint work with Abramsky, Hyttinen and Paolini).
- A very logical form of Bell's Theorem in quantum foundations (Hyttinen-Paolini 2014).

$$\forall\vec{x}\exists\vec{y}R(\vec{x},\vec{y}) \wedge [\forall\vec{x}\forall\vec{y}(R(\vec{x},\vec{y}) \rightarrow$$
$$\forall\vec{u}\exists\vec{v}(R(\vec{u},\vec{v}) \wedge (\vec{x} = \vec{u} \rightarrow \vec{y} = \vec{v})\wedge$$
$$\bigwedge_{i=1}^{n} {=}(\vec{x}\vec{y}u_i, v_i))]$$

Punchline

- The emergent logic of dependence and independence provides a common mathematical basis for fundamental concepts in biology, social science, physics, mathematics and computer science.
- We can find fundamental principles governing this logic.
- Algorithmic results show—as can be expected—that dependence logic has higher complexity than ordinary first order (propositional, modal) logic.
- Important parts can be completely axiomatized, other parts are manifestly beyond the reach of axiomatization.

Thank you!