The Expressive Power of Modal Dependence Logic

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Motivation and history

Logical modelling of uncertainty, imperfect information and functional dependence in the framework of modal logic.

The ideas are transfered from first-order dependence logic (and independence-friendly logic) to modal logic.

Historical development:

- Branching quantifiers by Henkin 1959.
- Compositional semantics for independence-friendly logic by Hodges 1997. (Origin of team semantics.)
- IF modal logic by Tulenheimo 2003.
- Dependence logic by Väänänen 2007.
- Modal dependence logic by Väänänen 2008.
Motivation and history

In IF modal logic, diamonds can be slashed by boxes that precede them: $\square_1(\Diamond_2/\square_1)\varphi$.

The idea in modal dependence logic ($\mathcal{MDL}$) is quite different than in IF modal logic: dependences are not between states, but truth values of propositions.

$\mathcal{MDL}$ is not able to express temporal dependencies; to remedy this, Ebbing et al. 2013 introduced extended modal dependence logic ($\mathcal{EMDL}$).

Propositional dependence logic is closely related to the *Inquisitive logic* of Groenendijk 2007.
Syntax for modal logic

**Definition**

Let $\Phi$ be a set of atomic propositions. The set of formulae for standard modal logic $\mathcal{ML}(\Phi)$ is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid (\varphi \lor \varphi) \mid (\varphi \land \varphi) \mid \Diamond \varphi \mid \square \varphi,$$

where $p \in \Phi$.

Note that formulas are assumed to be in negation normal form: negations may occur only in front of atomic formulas.
Kripke structures

Definition

Let $\Phi$ be a set of atomic propositions. A Kripke model $K$ over $\Phi$ is a tuple $K = (W, R, V)$, where $W$ is a nonempty set of worlds, $R \subseteq W \times W$ is a binary relation, and $V$ is a valuation $V : \Phi \to P(W)$. 
Kripke semantics for $\mathcal{ML}$ is defined as follows.

\[ K, w \models p \iff w \in V(p). \]
\[ K, w \models \neg p \iff w \notin V(p). \]
\[ K, w \models \varphi \lor \psi \iff K, w \models \varphi \text{ or } K, w \models \psi. \]
\[ K, w \models \varphi \land \psi \iff K, w \models \varphi \text{ and } K, w \models \psi. \]
\[ K, w \models \Diamond \varphi \iff K, w' \models \varphi, \text{ for some } w' \text{ s.t. } xRw'. \]
\[ K, w \models \square \varphi \iff K, w \models \varphi, \text{ for all } w' \text{ s.t. } xRw'. \]
Team semantics?

1. In this context a team is a set of possible worlds, i.e., if $K = (W, R, V)$ is a Kripke model then $T \subseteq W$ is a team of $K$.

2. The standard semantics for modal logic is given with respect to pointed models $K, w$. In team semantics the semantics is given for models and teams, i.e., with respect to pairs $K, T$, where $T$ is a team of $K$.

3. Some possible interpretations for $K, w$ and $K, T$:
   (a) $K, w \models \phi$: The actual world is $w$ and $\phi$ is true in $w$.
   (b) $K, T \models \phi$: The actual world is in $T$, but we do not know which one it is. The formula $\phi$ is true in the actual world.
   (c) $K, T \models \phi$: We consider sets of points as primitive. The formula $\phi$ describes properties of collections of points.
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Background

Modal logic

Team semantics

Modal dependence logic

Modal definability

Succinctness

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Team semantics for modal logic

Definition

Kripke/Team semantics for $\mathcal{ML}$ is defined as follows. Remember that $K = (W, R, V)$ is a normal Kripke model and $T \subseteq W$.

- $K, w \models p \iff w \in V(p)$.
- $K, w \models \neg p \iff w \notin V(p)$.
- $K, w \models \varphi \land \psi \iff K, w \models \varphi$ and $K, w \models \psi$.
- $K, w \models \varphi \lor \psi \iff K, w \models \varphi$ or $K, w \models \psi$.
- $K, w \models \Box \varphi \iff K, w' \models \varphi$ for every $w'$ s.t. $wRw'$.
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- $K, T \models \Diamond \varphi$ $\iff$ $K, T' \models \varphi$ for some $T'$ s.t. $\forall w \in T \exists w' \in T' : wRw'$ and $\forall w' \in T' \exists w \in T : wRw'$.

Note that $K, \emptyset \models \varphi$ for every formula $\varphi$. 
Team semantics vs. Kripke semantics

**Theorem (Flatness property of ML)**

Let $K$ be a Kripke model, $T$ a team of $K$ and $\varphi$ a $\mathcal{ML}$-formula. Then

\[ K, T \models \varphi \iff K, w \models \varphi \text{ for all } w \in T, \]

in particular

\[ K, \{w\} \models \varphi \iff K, w \models \varphi. \]

Note that it also follows that every $\mathcal{ML}$-formula is *downwards closed*:

If $K, T \models \varphi$, then $K, S \models \varphi$ for all $S \subseteq T$. 
Modal dependence logic

Introduced by Väänänen 2008, the syntax modal dependence logic $\mathcal{MDL}$ extends the syntax of modal logic by the clause

$$\text{dep}(p_1, \ldots, p_n, q),$$

where $p_1, \ldots, p_n, q$ are proposition symbols.
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The intended meaning of the atomic formula

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Semantics for MDL

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Semantics for $\mathcal{MDL}$

The intended meaning of the atomic formula

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is that the truth value of the propositions $p_1, \ldots, p_n$ functionally determines the truth value of the proposition $q$.

The semantics for $\mathcal{MDL}$ extends the semantics of $\mathcal{ML}$, defined with teams, by the following clause:

$$K, T \models \text{dep}(p_1, \ldots, p_n, q)$$

if and only if $\forall w_1, w_2 \in T$:

$$\bigwedge_{i \leq n} (w_1 \in V(p_i) \iff w_2 \in V(p_i)) \Rightarrow (w_1 \in V(q) \iff w_2 \in V(q)).$$
Intuitionistic disjunction

\[\mathcal{ML}(\otimes): \text{add a different version of disjunction} \otimes \text{ to modal logic with the semantics:} \]

\[K, T \models \varphi \otimes \psi \iff K, T \models \varphi \text{ or } K, T \models \psi.\]

Dependence atoms are definable in \(\mathcal{ML}(\otimes)\) (Väänänen 09):

\[K, T \models \text{dep}(p_1, \ldots, p_n, q) \iff K, T \models \bigvee_{s \in F}(\theta_s \land (q \otimes \neg q)),\]

where \(F\) is the set of all \(\{p_1, \ldots, p_n\}\)-assignments, and \(\theta_s\) is the formula \(\bigwedge_{i \leq n} p_s^{s(p_i)}\), where \(p_i^\bot = \neg p_i\) and \(p_i^\top = p_i\).
Intuitionistic disjunction

It is easy to prove by induction that for every $\mathit{MDL}$-formula there is an equivalent $\mathit{ML}(\otimes)$-formula.

Thus, $\mathit{MDL} \leq \mathit{ML}(\otimes)$.

However, the converse is not true: There is no formula $\varphi \in \mathit{MDL}$ that is equivalent with $\lozenge p \otimes \Box \neg p$.

Thus, $\mathit{MDL} < \mathit{ML}(\otimes)$. 
Extended modal dependence logic $\text{EMDL}$

What is missing from $\text{MDL}$? The counterexample gives a clue: the formula $\Diamond p \otimes \Box \neg p$ is equivalent to $\text{dep}(\Diamond p)$. Thus, we need dependencies between arbitrary modal formulas.

$\text{EMDL}(\Phi)$-formulas are defined by the following grammar:

$$\varphi ::= p \mid \neg p \mid \text{dep}(\psi_1, \ldots, \psi_n, \theta) \mid (\varphi \lor \varphi) \mid (\varphi \land \varphi) \mid \Box \varphi \mid \Diamond \varphi,$$

where $p \in \Phi$ and $\psi_1, \ldots, \psi_n, \theta \in \text{ML}$.

The semantics of $\text{dep}(\psi_1, \ldots, \psi_n, \theta)$ is given as for $\text{dep}(p_1, \ldots, p_n, q)$.

With these more general dependence atoms we can express for example temporal dependencies.
Properties of $\mathsf{EMDL}$

Using the idea of Väänänen 09, we can prove that $\mathsf{EMDL}$ is contained in $\mathcal{ML}(\otimes)$:

**Theorem** (Ebbing, Hella, Meier, Müller, V., Vollmer 13)

$\mathcal{MDL} < \mathsf{EMDL} = \mathcal{ML}(\otimes\mathcal{ML}) \leq \mathcal{ML}(\otimes)$.

($\mathcal{ML}(\otimes\mathcal{ML})$ is the syntactic fragment of $\mathcal{ML}(\otimes)$ in which the clause $\varphi \otimes \varphi$ is applied only to $\mathcal{ML}$-formulae.)

All these logics are downward closed:

**Theorem**

Let $\varphi \in \mathcal{ML}(\otimes)$. If $K, T \models \varphi$, then $K, S \models \varphi$ for all $S \subseteq T$. 
Modal definability and bisimulation

Let $\equiv_k$ denote the usual $k$-bisimulation for modal logic.

A class $C$ of pointed Kripke models $(K, w)$ is closed under $k$-bisimulation if it satisfies the condition:

- $(K, w) \in C$ and $K, w \equiv_k K', w'$ implies that $(K', w') \in C$. 

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It is well-known that modal definability can be characterized in terms of closure under $k$-bisimulation:

**Theorem (Gabbay, van Benthem)**

A class $C$ of pointed Kripke models is definable in $\mathcal{ML}$ if and only if $C$ is closed under $k$-bisimulation for some $k \in \mathbb{N}$. 
**Team bisimulation**

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $(K, T), (K', T')$ Kripke models with teams and $k \in \mathbb{N}$. Then $K, T$ and $K', T'$ are <strong>team $k$-bisimilar</strong>, $K, T \xleftrightarrow{k} K', T'$, if</td>
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We say that a class $\mathcal{C}$ of Kripke models with teams is **closed under team $k$-bisimulation** if it satisfies the condition:

$\triangleright (K, T) \in \mathcal{C}$ and $K, T \xleftrightarrow{k} K', T'$ implies that $(K', T) \in \mathcal{C}$. 
The expressive power of $\mathcal{ML}(\otimes)$

Theorem (Hella, Luosto, Sano, V. 14)

A class $\mathcal{C}$ is definable in $\mathcal{ML}(\otimes)$ if and only if $\mathcal{C}$ is downward closed and there exists $k \in \mathbb{N}$ such that $\mathcal{C}$ is closed under team $k$-bisimulation.
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**Theorem (Hella, Luosto, Sano, V. 14)**

A class $\mathcal{C}$ is definable in $\mathcal{ML}(\otimes)$ if and only if $\mathcal{C}$ is downward closed and there exists $k \in \mathbb{N}$ such that $\mathcal{C}$ is closed under team $k$-bisimulation.

This result is a natural fusion of the Gabbay – van Benthem characterization for $\mathcal{ML}$, and a corresponding result for the propositional fragment $\mathcal{PL}(\otimes)$ of $\mathcal{ML}(\otimes)$:

**Theorem (Ciardelli 09, Yang 14)**

All downward closed properties of propositional teams are definable in $\mathcal{PL}(\otimes)$. 
The expressive power of $\mathcal{EMDL}$

Remember that $\mathcal{EMDL} \leq \mathcal{ML}(\emptyset)$.

Theorem (Hella, Luosto, Sano, V. 14)

$\mathcal{ML}(\emptyset) \leq \mathcal{EMDL}$. Consequently, $\mathcal{EMDL} \equiv \mathcal{ML}(\emptyset)$. 
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$\mathcal{ML}(\emptyset) \leq \mathcal{EMDL}$. Consequently, $\mathcal{EMDL} \equiv \mathcal{ML}(\emptyset)$.

Corollary

$\mathcal{ML}(\emptyset) \equiv \mathcal{ML}(\emptyset \mathcal{ML})$.

Corollary

A class $C$ is definable in $\mathcal{EMDL}$ iff $C$ is downward closed and there exists $k \in \mathbb{N}$ s.t. $C$ is closed under team $k$-bisimulation.
EMDL is exponentially more succinct than $\mathcal{ML}(\forall)$

**Theorem** (Hella, Luosto, Sano, V. 14)

Let $\varphi$ be a formula of $\mathcal{ML}(\forall)$ that is equivalent with $\text{dep}(p_1, \ldots, p_n, q)$. Then $|\varphi| > 2^n$. 
Thanks!
Bibliography
