A nonparametric multivariate multisample test

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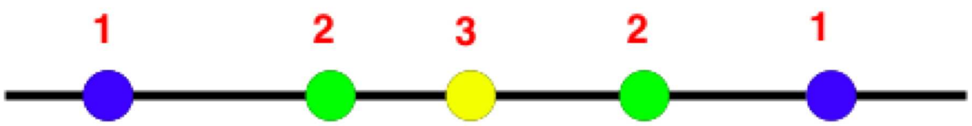
Motivation

- Need for Ordering and Ranking of Data:
  - Natural interest in Ordering and Ranking
    - Extremeness (e.g. maximum flood levels)
    - Variability (e.g. Range)
    - Effects of external contamination
  - Exploitation of order for inference:
    - Outlier Detection
    - Robustness
    - Non-parametric methods based on ranks and signs
    - Probability Plots for model validation
- How could we order the multivariate data?
- The problem was initially started by defining multidimensional medians.

Halfspace Depth (HSD)



- For $x_1, \ldots, x_n \in \mathbb{R}$ and $x \in \mathbb{R}$

$$HSD(x;\, F_n) = \frac{\min\{\#\{i;\, x_i \le x\},\, \#\{i;\, x_i \ge x\}\}}{n}$$

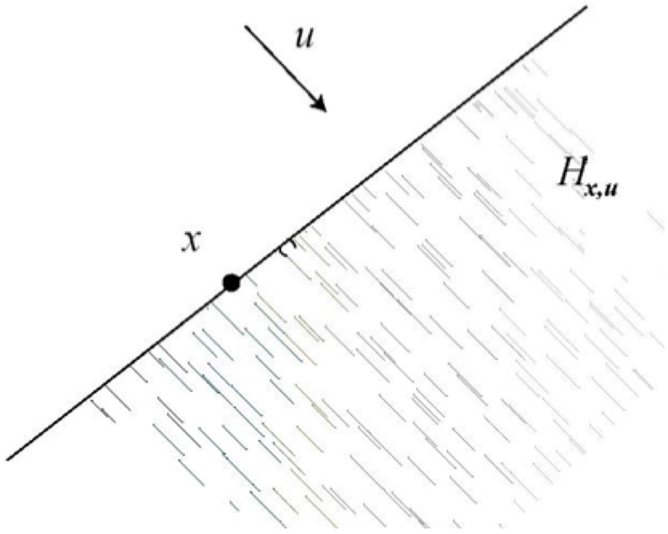
- For $x \in \mathbb{R}$ and distribution F

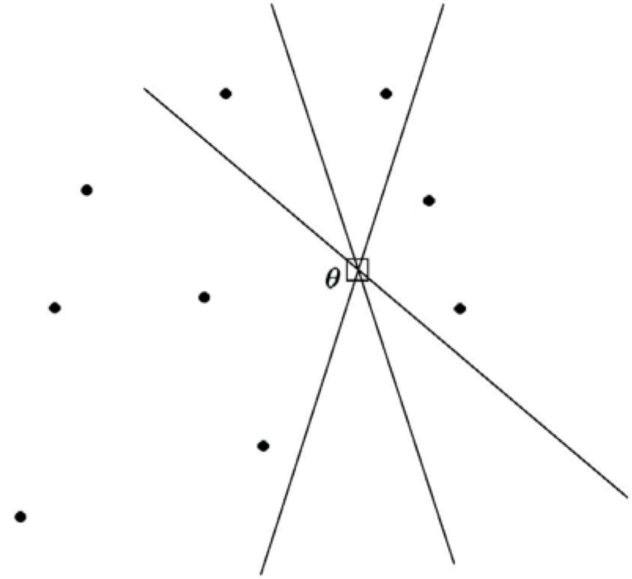
$$\begin{aligned} HSD(x;\, F) &= \min\{P(X \le x),\, P(X \ge x)\} \\ &= \min\{F(x),\, 1 - F(x^-)\} \end{aligned}$$

- Median of F is $\quad \arg\max_{x \in \mathbb{R}} HSD(x;\, F)$

Halfspace Depth (HSD)

- Hodges (1955), Tukey (1975), and Donoho and Gasko (1992)





- For $\mathbf{X}_1, \ldots, \mathbf{X}_n \sim F$ and $\mathbf{x} \in \mathbb{R}^d$

$$HSD(\mathbf{x}, F_n) = HSD(\mathbf{x}, \mathbf{X}) = n^{-1} \min_{||\mathbf{u}||=1} \#\{i \ : \ \mathbf{u}'\mathbf{X}_i \geq \mathbf{u}'\mathbf{x}\}$$

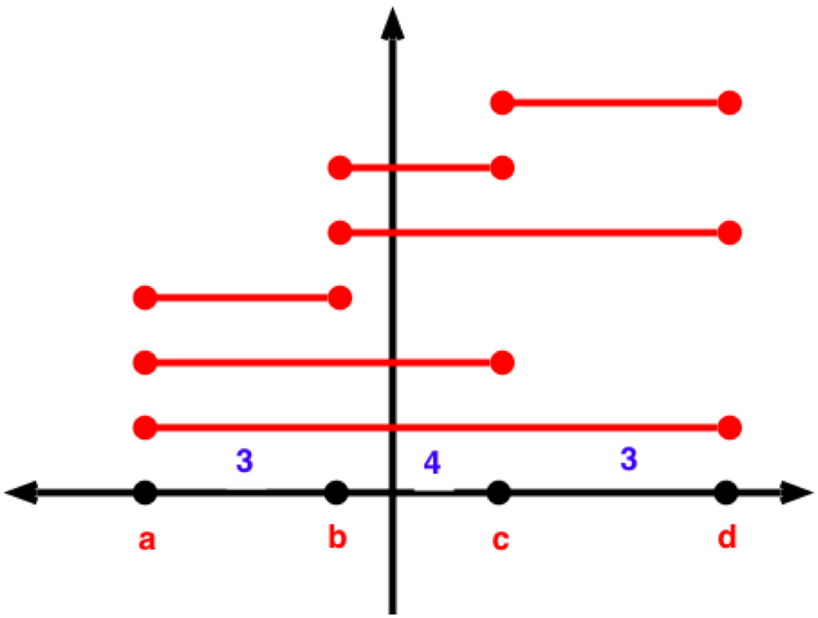
- For $\mathbf{x} \in \mathbb{R}^d$ and d-dimensional distribution F

$$HSD(\mathbf{x}; F) = \inf_{||\mathbf{u}||=1} P_F(\mathbf{u}^T\mathbf{X} \geq \mathbf{u}^T\mathbf{x})$$

Halfspace Depth (HSD)

- HS-Median $= \arg\max_{\mathbf{x}\in\mathbb{R}^d} HSD(\mathbf{x}, F_n)$ is $\sqrt{n}$-consistent (Nolan 1999, Bai and He 1999), and has asymptotic breakdown point $1/3$ (Donoho and Gasko 1992), and a bounded influence function (Chen and Tyler 2002).
- The empirical HSD converges to the theoretical one almost surly (Donoho and Gasko 1992), has a bounded influence function (Romanazzi 2002), and determines the empirical distribution (Struyf and Rousseeuw 1999). HSD characterizes distributions uniquely (Koshevoy 2002, Koshevoy 200x),
- Algorithms for exact calculation of HSD in $\mathbb{R}^d$ has $O(n^{d-1}log(n))$, an approximate algorithm has $O(md^3 + mdn)$ time, where m is number of d-subsets used. (Rousseeuw and Ruts 1996, 1998, Ruts and Rousseeuw 1996, Rousseeuw and Struyf 1998), see also Aloupis et al (2002, 2003), Chakraborty and Chaudhuri (2003), Chan (2004).

Simplicial Depth (SD)



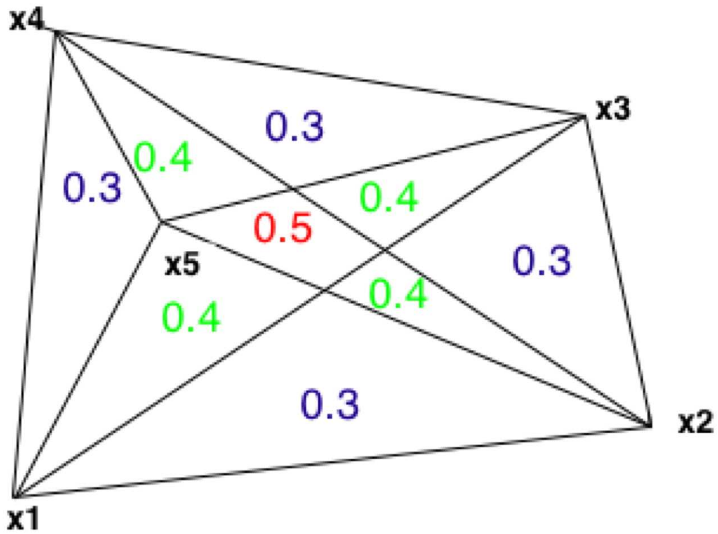
- For $x_1, \ldots x_n \in \mathbb{R}$, $\quad SD(x;\, F_n) = \binom{n}{2}^{-1} \sum I([x_{i_1},\, x_{i_2}] \ni x)$
- For X_1, X_2 from F and $x \in \mathbb{R}$

$$SD(x;\, F) = 2P(X_1 \le x \le X_2) = 2F(x)(1 - F(x))$$

- Median of F is $\quad \arg\max_{x \in \mathbb{R}} SD(x;\, F)$

Simplicial Depth (SD)

- Liu (1988) and Liu (1990),



- For $\mathbf{x}_1, \ldots \mathbf{x}_n \in \mathbb{R}^d$,

$$SD(\mathbf{x};\, F_n) = \binom{n}{d+1}^{-1} \sum I(S[\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_2}] \ni \mathbf{x})$$

- For $\mathbf{X}_1, \ldots, \mathbf{X}_2$ from F and $\mathbf{x} \in \mathbb{R}^d$,

$$SD(\mathbf{x};\, F) = P(S[\mathbf{X}_1, \ldots, \mathbf{X}_2] \ni \mathbf{x})$$

Simplicial Depth (SD)

- S-Median $= \arg\max_{\mathbf{x}\in\mathbb{R}^d} SD(\mathbf{x}, F_n)$ is strongly consistent (Liu 1990, and Arcones and Giné 1994), has positive breakdown point (Chen 1995). It also has an asymptotic normal distribution (Arcones, Chen and Giné 1994), which proof uses U-process approach.
- The empirical SD converges to the theoretical one almost surly (Liu 1990, D umbgen 1990, and Arcones and Giné 1993).
- An algorithm for exact calculation of SD in 2-dimensional space has $O(nlog(n))$ time complexity (Rousseeuw and Ruts 1996). More computational issues and algorithms are discussed in Aloupis (2001), Aloupis et al (2002, 2003), Burr et al (2003) Cheng and Ouyang (2000), Gil et al (1992), and Huang and Small (2004).

Spatial Depth (SPD)

- In dimension 1, the median may be characterized as

$$\text{Sample Median} = \arg\min_{\theta \in \mathbb{R}} \sum_{i=1}^{n} |x_i - \theta|$$

$$\text{Population Median} = \arg\min_{\theta \in \mathbb{R}} E|X - \theta|$$

- SPD of $\mathbf{x} \in \mathbb{R}^d$ is defined by

$$SPD(\mathbf{x}, F) = \frac{1}{1 + E||\mathbf{x} - \mathbf{X}||}$$

- The empirical version is:

$$SPD(\mathbf{x}, F_n) = \frac{1}{1 + \sum_{i=1}^{n} ||\mathbf{x} - \mathbf{X}_i||}$$

Spatial Depth (SPD)

- A fast algorithm for computing the spatial median is given by Vardi and Zhang (2000)
- The asymptotics of the spatial median are discussed by Brown (1983), Pollard (1984).
- The spatial median has 50% breakdown point, (Kemperman 1987, and Lopuha a and Rousseeuw 1991), and as an M-estimator, bounded influence function.
- Affine invariant version of SPD can be

$$ASPD(\mathbf{x}, F) = \frac{1}{1 + E||\mathbf{x} - \mathbf{X}||_{\Sigma^{-1}}}$$

- where $\quad ||\mathbf{x}||_M = \sqrt{\mathbf{x}^T M \mathbf{x}}, \quad \forall \mathbf{x} \in \mathbb{R}^d.$

Mahalanobis Depth (MHD)

- Mahalanobis distance; Mahalanobis (1936)
- For $\mathbf{x} \in \mathbb{R}^d$ and F with mean $\mu(F)$ and covariance $\Sigma(F)$,

$$MHD(\mathbf{x}; F) = \frac{1}{1 + (\mathbf{x} - \mu_{(F)})' \Sigma^{-1}_{(F)} (\mathbf{x} - \mu_{(F)})}$$

- For the empirical version, replace $\mu(F)$ and $\Sigma(F)$ with appropriate estimates.
- To robustify MHD, we can use MCD or MVE estimators.
- We suggest using MCD, because of nice asymptotics (Butler, Davies, and Jhun 1993)

Other Depth functions

- Convex Hull Peeling Depth (Barnett 1976, Green 1982)
- Projection Depth (Stahel 1981, Donoho 1982, Zuo & Serfling 2000)
- Oja's Simplicial Volume Depth (Oja 1983, Zuo & Serfling 2000)
- Majority Depth (Singh 1991)
- Likelihood Depth (Fraiman and Meloche 1996),
- Zonoid Depth (Koshevoy & Mosler 1997)
- Regression Depth (Rousseeuw & Hubert 1999)
- Review on several multidimensional medians and depth functions (Small (1990), and Chenouri 2004).

Depth based Ordering

- Compute the depths of $\mathbf{X}_1, ..., \mathbf{X}_n,$

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  - $\mathbf{X}_{[1]}$ being the deepest or the most central point.
  - $\mathbf{X}_{[n]}$ being the most *outlying point*.

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- $R(\mathbf{X}_i)$ is the linear rank of $D(\mathbf{X}_i, F_n)$ w.r.t. the set

$$\{D(\mathbf{X}_1, F_n), \ldots, D(\mathbf{X}_n, F_n)\}$$

Multivariate Dispersion Tests

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Treatments

1	2	$\ldots$	t
$\mathbf{X}_{11}$	$\mathbf{X}_{12}$	$\ldots$	$\mathbf{X}_{1t}$
$\mathbf{X}_{21}$	$\mathbf{X}_{22}$	$\ldots$	$\mathbf{X}_{2t}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathbf{X}_{n_1 1}$	$\mathbf{X}_{n_2 2}$	$\ldots$	$\mathbf{X}_{n_t t}$

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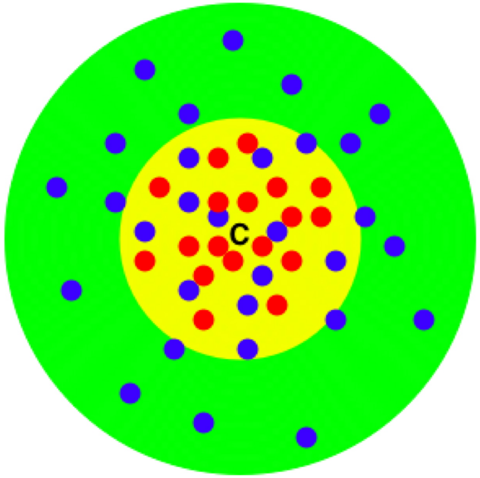

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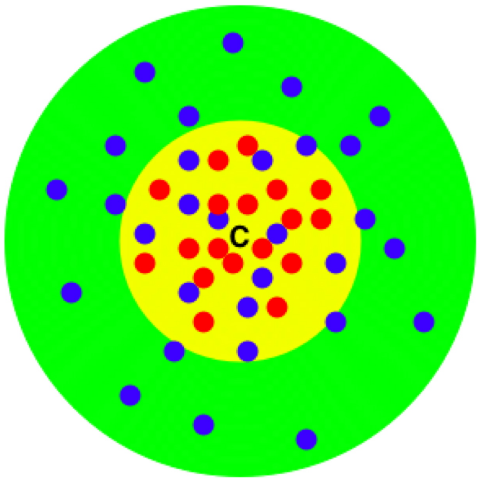

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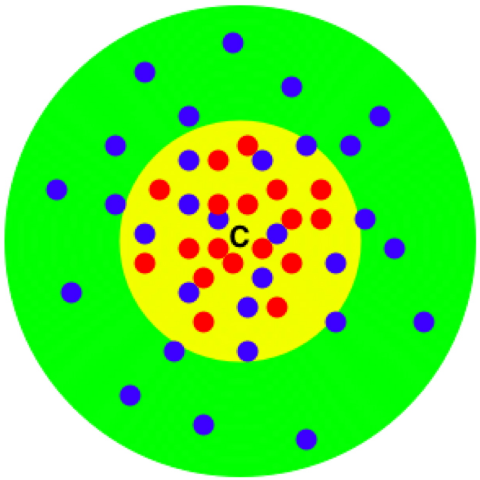

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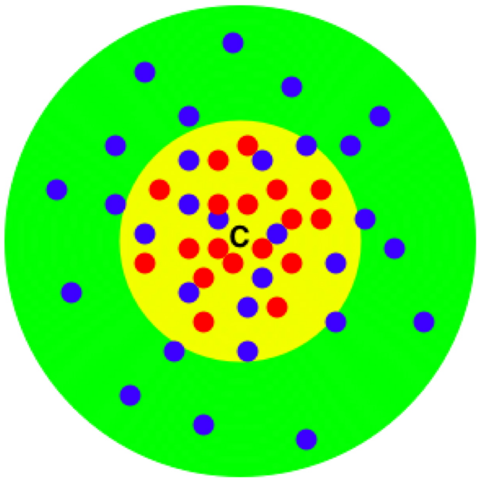

- Choose a depth function
- Calculate the depths of the pooled data w.r.t to the pooled data
- Find the center-outward ranks of the data

Multivariate Dispersion Tests

Treatments

1	2	$\ldots$	t
R_{11}	R_{12}	$\ldots$	R_{1t}
R_{21}	R_{22}	$\ldots$	R_{2t}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$R_{n_1 1}$	$R_{n_2 1}$	$\ldots$	$R_{n_t t}$
$\bar{R}_{\cdot 1}$	$\bar{R}_{\cdot 2}$	$\ldots$	$\bar{R}_{\cdot t}$

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$R_{n_1 1}$	$R_{n_2 1}$	…	$R_{n_t t}$
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- The test statistic is

$$K = \sum_{j=1}^{t} \left(1 - \frac{n_j}{n}\right) \frac{12 n_j}{(n - n_j)(n+1)} \left(\bar{R}_{\cdot j} - \frac{n+1}{2}\right)^2$$

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- K is distribution free, & its asymptotic distribution is $\chi^2(k-1)$.

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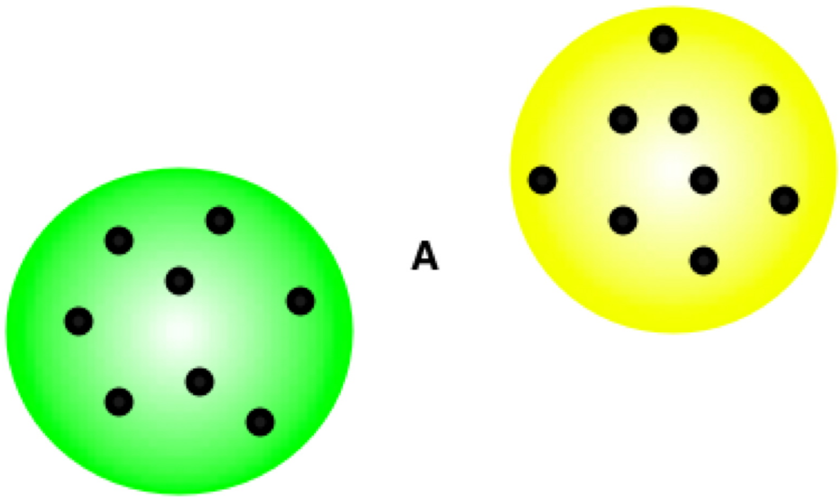

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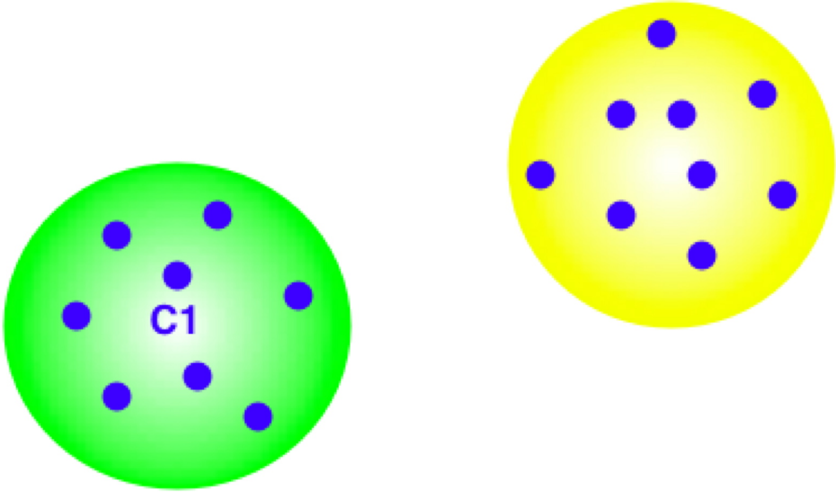

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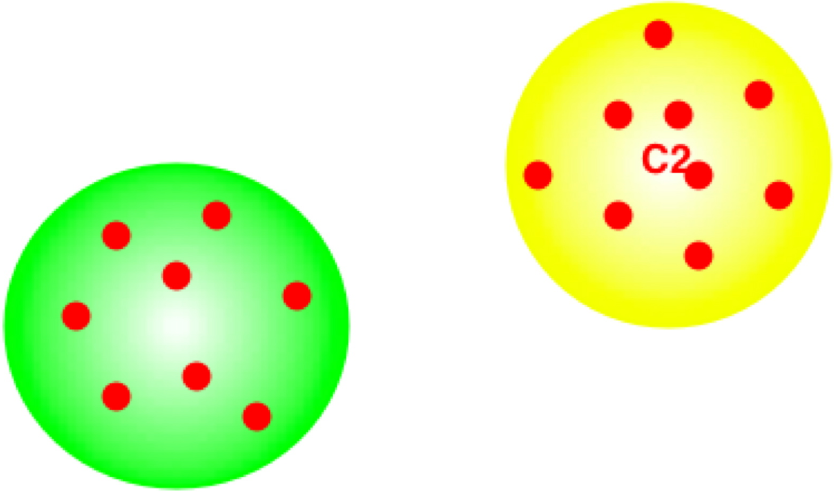

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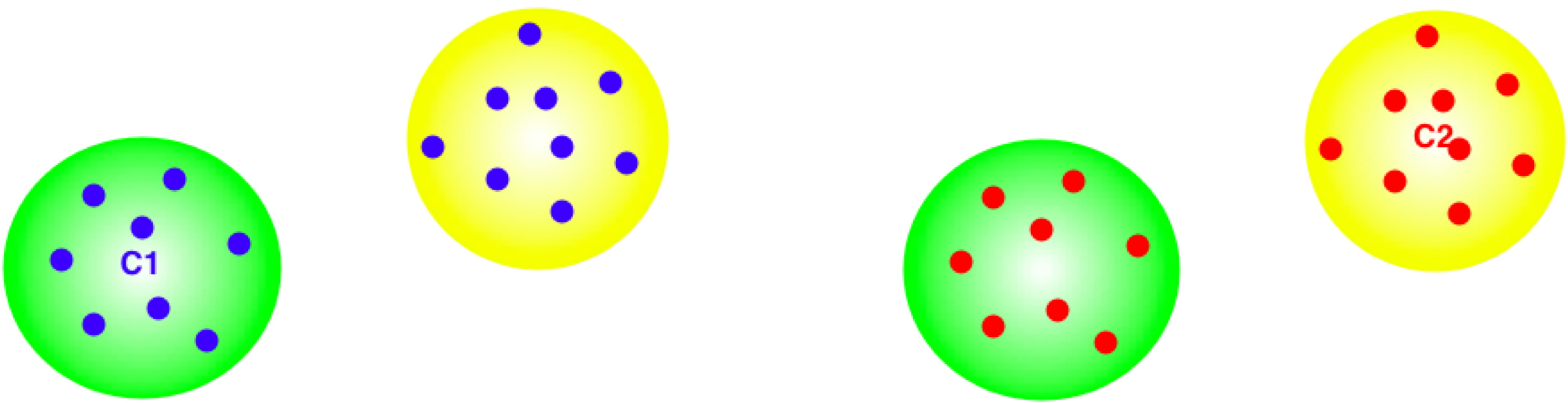

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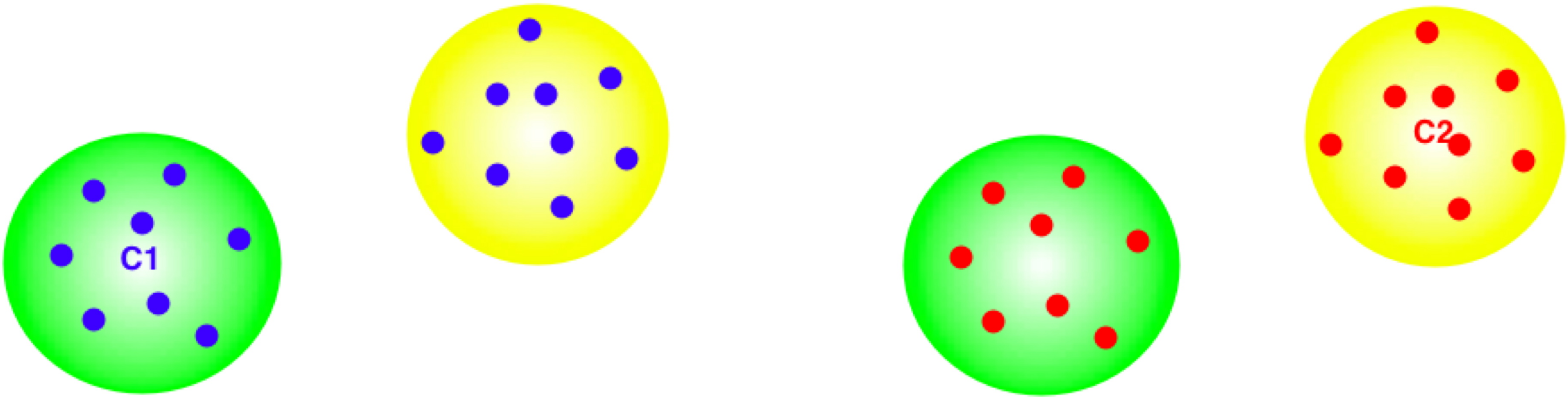

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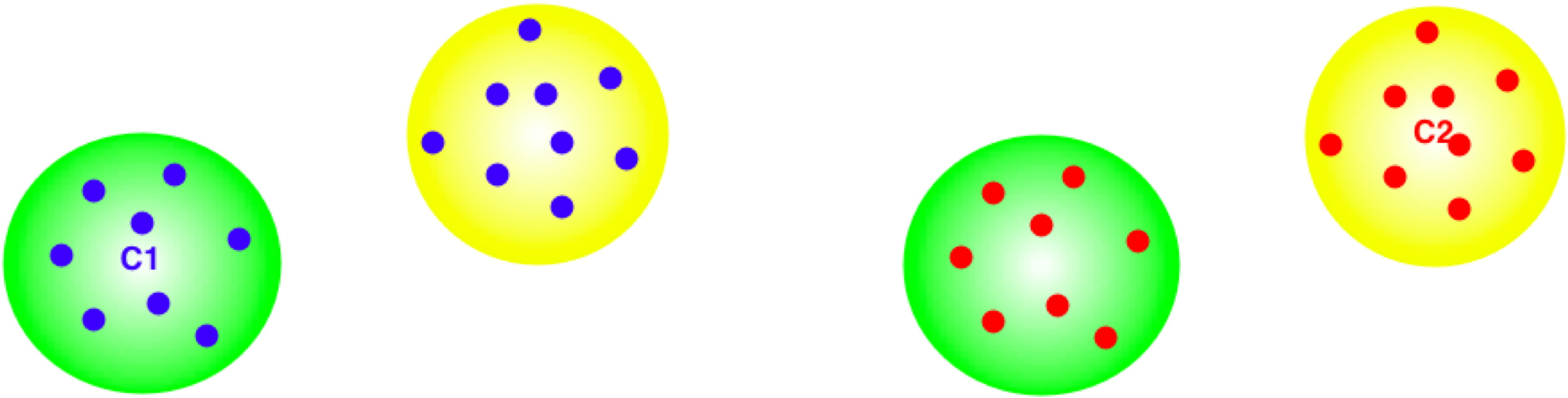

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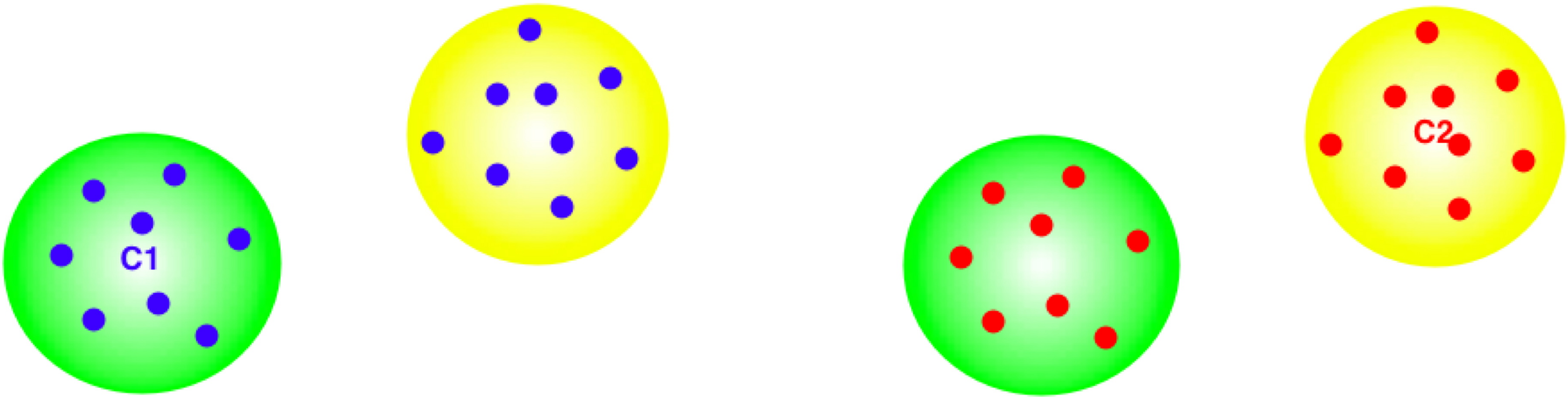

- Choose a depth function
- Calculate the depths of the pooled data w.r.t to each data set
- Find the center-outward ranks of the pooled data in each k tables

Multivariate Dispersion Tests

Treatments			
1	2	$\dots$	t
$R_{11}(k)$	$R_{12}(k)$	$\dots$	$R_{1t}(k)$
$R_{21}(k)$	$R_{22}(k)$	$\dots$	$R_{2t}(k)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$R_{n_1 1}(k)$	$R_{n_2 1}(k)$	$\dots$	$R_{n_t t}(k)$
$\bar{R}_{\cdot 1}(k)$	$\bar{R}_{\cdot 2}(k)$	$\dots$	$\bar{R}_{\cdot t}(k)$

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$\bar{R}_{\cdot 1}(k)$	$\bar{R}_{\cdot 2}(k)$	$\dots$	$\bar{R}_{\cdot t}(k)$

- $$H(k) = \sum_{j=1}^{t} \left(1 - \frac{n_j}{n}\right) \frac{12 n_j}{(n-n_j)(n+1)} \left(\bar{R}_{\cdot j}(k) - \frac{n+1}{2}\right)^2$$

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- $H(k) \xrightarrow{\mathcal{D}} \chi^2(t-1)$ and for $H(k) - H(k') \xrightarrow{a.s.} 0$

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- $H(k) = \sum_{j=1}^{t} \left(1 - \frac{n_j}{n}\right) \frac{12 n_j}{(n-n_j)(n+1)} \left(\bar{R}_{\cdot j}(k) - \frac{n+1}{2}\right)^2$
- $H(k) \xrightarrow{\mathcal{D}} \chi^2(t-1)$ and for $H(k) - H(k') \xrightarrow{a.s.} 0$
- The test statistics H given below has an asymptotic $\chi^2(t-1)$

$$H = \frac{1}{t} \sum_{k=1}^{t} H(k)$$

Simulations for Approximating

$$P_{H_0}(\text{rejecting} H_0)$$

1000 samples of size $n_1 = n_2 = 100$, $\alpha = 0.05$.

			Our Proposed Tests H		
Distributions	T^2	SPKW	RMHD	RASPD	HSD
$\mathbf{X} \sim N(\mathbf{0}, \Sigma)$, $\mathbf{Y} \sim N(\mathbf{0}, \Sigma)$	0.051	0.050	0.049	0.049	0.080

Where $\Sigma = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}$

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Distributions	T^2	SPKW	RMHD	RASPD	HSD
$\mathbf{X} \sim N(\mathbf{0}, \Sigma)$, $\mathbf{Y} \sim N(\mathbf{0}, \Sigma)$	0.051	0.050	0.049	0.049	0.080
$\mathbf{X} \sim 0.5N(\mathbf{0}, \Sigma) + 0.5N(\mathbf{6}, \Sigma)$ $\mathbf{Y} \sim 0.5N(\mathbf{0}, \Sigma) + 0.5N(\mathbf{6}, \Sigma)$	0.052	0.055	0.055	0.057	0.085

Where $\Sigma = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}$

Simulations for Approximating

$P_{H_0}(\text{rejecting}\, H_0)$

1000 samples of size $n_1 = n_2 = 100$, $\alpha = 0.05$.

			Our Proposed Tests H		
Distributions	T^2	SPKW	RMHD	RASPD	HSD
$\mathbf{X} \sim N(\mathbf{0}, \Sigma)$, $\mathbf{Y} \sim N(\mathbf{0}, \Sigma)$	0.051	0.050	0.049	0.049	0.080
$0.5N(\mathbf{0}, \Sigma) + 0.5N(\mathbf{6}, \Sigma)$ $0.5N(\mathbf{0}, \Sigma) + 0.5N(\mathbf{6}, \Sigma)$	0.052	0.055	0.055	0.057	0.085
$\mathbf{X} \sim C(0,1) \times C(0,1)$ $\mathbf{Y} \sim C(0,1) \times C(0,1)$	0.017	0.046	0.043	0.045	0.065

Where $\Sigma = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}$

Simulations for Approximating

$$P_{H_1}(\text{rejecting}\, H_0)$$

			Our Proposed Tests H		
Distributions	T^2	SPKW	RMHD	RASPD	HSD
$\mathbf{X} \sim N(\mathbf{0}, \Sigma)$, $\mathbf{Y} \sim N(\mathbf{1}, \Sigma)$	1	1	0.972	0.986	0.998

Where $\Sigma = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}$, and $\Sigma^* = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$

Simulations for Approximating

$$P_{H_1}(\text{rejecting}\, H_0)$$

			Our Proposed Tests H		
Distributions	T^2	SPKW	RMHD	RASPD	HSD
$\mathbf{X} \sim N(\mathbf{0}, \Sigma)$, $\mathbf{Y} \sim N(\mathbf{1}, \Sigma)$	1	1	0.972	0.986	0.998
$\mathbf{X} \sim N(\mathbf{0}, \Sigma)$, $\mathbf{Y} \sim N(\mathbf{0}, \Sigma^*)$	0.051	0.056	1	1	1

Where $\Sigma = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}$, and $\Sigma^* = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$

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Distributions	T^2	SPKW	RMHD	RASPD	HSD
$\mathbf{X} \sim N(\mathbf{0}, \Sigma),\ \mathbf{Y} \sim N(\mathbf{1}, \Sigma)$	1	1	0.972	0.986	0.998
$\mathbf{X} \sim N(\mathbf{0}, \Sigma),\ \mathbf{Y} \sim N(\mathbf{0}, \Sigma^*)$	0.051	0.056	1	1	1
$\mathbf{X} \sim 0.5N(\mathbf{0}, \Sigma) + 0.5N(\mathbf{6}, \Sigma)$ $\mathbf{Y} \sim 0.5N(\mathbf{2}, \Sigma) + 0.5N(\mathbf{8}, \Sigma)$	0.983	0.665	0.979	0.988	0.998

Where $\Sigma = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}$, and $\Sigma^* = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$

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$\mathbf{X} \sim N(\mathbf{0}, \Sigma),\ \mathbf{Y} \sim N(\mathbf{0}, \Sigma^*)$	0.051	0.056	1	1	1
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$\mathbf{X} \sim C(0,1) \times C(0,1)$ $\mathbf{Y} \sim C(0,1) \times C(5,1)$	0.503	1	1	1	1

Where $\Sigma = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}$, and $\Sigma^* = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$

Simulations for Approximating

$P_{H_1}(\text{rejecting}\, H_0)$

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Distributions	T^2	SPKW	RMHD	RASPD	HSD
$\mathbf{X} \sim N(\mathbf{0}, \Sigma),\ \mathbf{Y} \sim N(\mathbf{1}, \Sigma)$	1	1	0.972	0.986	0.998
$\mathbf{X} \sim N(\mathbf{0}, \Sigma),\ \mathbf{Y} \sim N(\mathbf{0}, \Sigma^*)$	0.051	0.056	1	1	1
$\mathbf{X} \sim 0.5N(\mathbf{0}, \Sigma) + 0.5N(\mathbf{6}, \Sigma)$ $\mathbf{Y} \sim 0.5N(\mathbf{2}, \Sigma) + 0.5N(\mathbf{8}, \Sigma)$	0.983	0.665	0.979	0.988	0.998
$\mathbf{X} \sim 0.5N(\mathbf{0}, \Sigma) + 0.5N(\mathbf{6}, \Sigma)$ $\mathbf{Y} \sim 0.5N(\mathbf{0}, \Sigma) + 0.5N(\mathbf{8}, \Sigma)$	0.633	0.658	0.969	0.988	0.997
$\mathbf{X} \sim C(0,1) \times C(0,1)$ $\mathbf{Y} \sim C(0,1) \times C(5,1)$	0.503	1	1	1	1
$\mathbf{X} \sim C(0,1) \times C(0,1)$ $\mathbf{Y} \sim C(0,0.4) \times C(0,0.4)$	0.009	0.053	1	1	1

Where $\Sigma = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}$, and $\Sigma^* = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$

Depth-Depth Plots

- Chenouri (2004), and Liu et al (1999)

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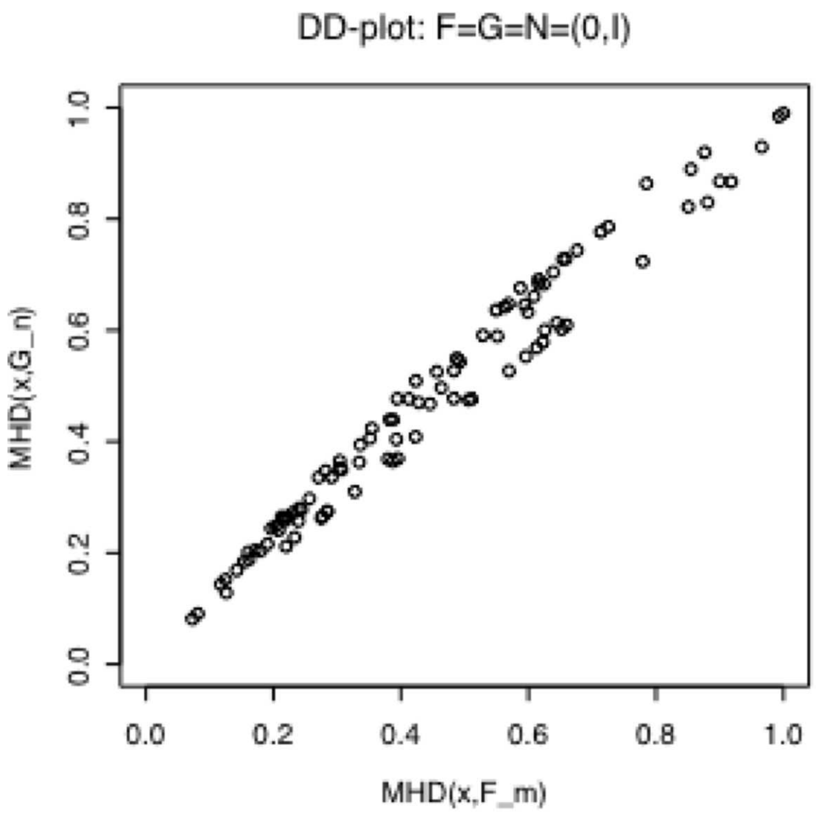
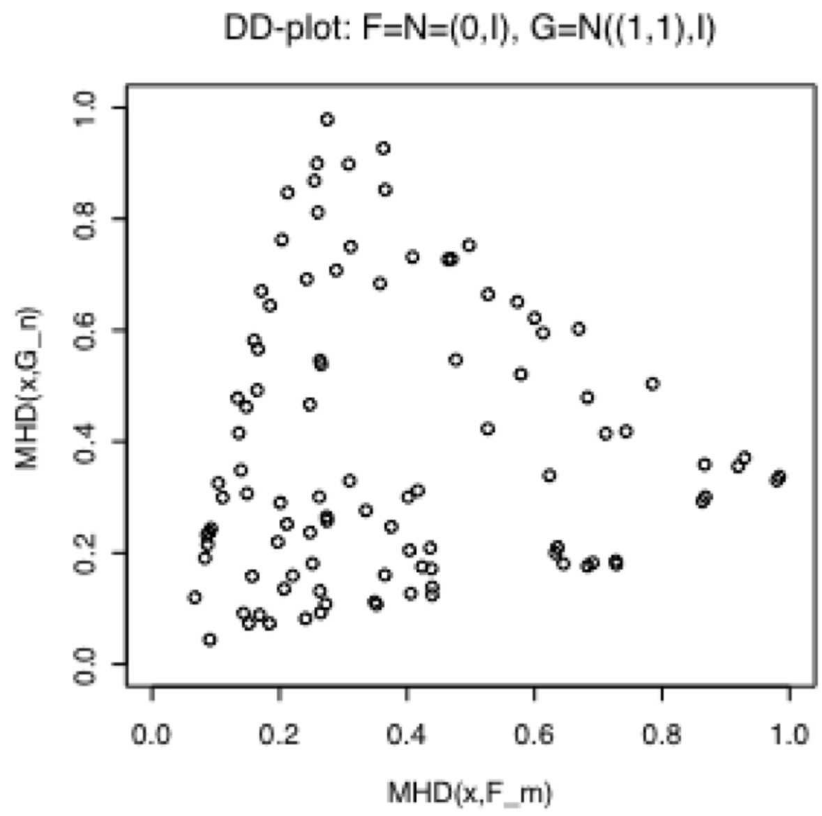
- If $F = G$, the DD-Plot and ranked DD-Plot should be concentrated along the line going through $(0,0),\ (1,1)$

Depth-Depth Plots





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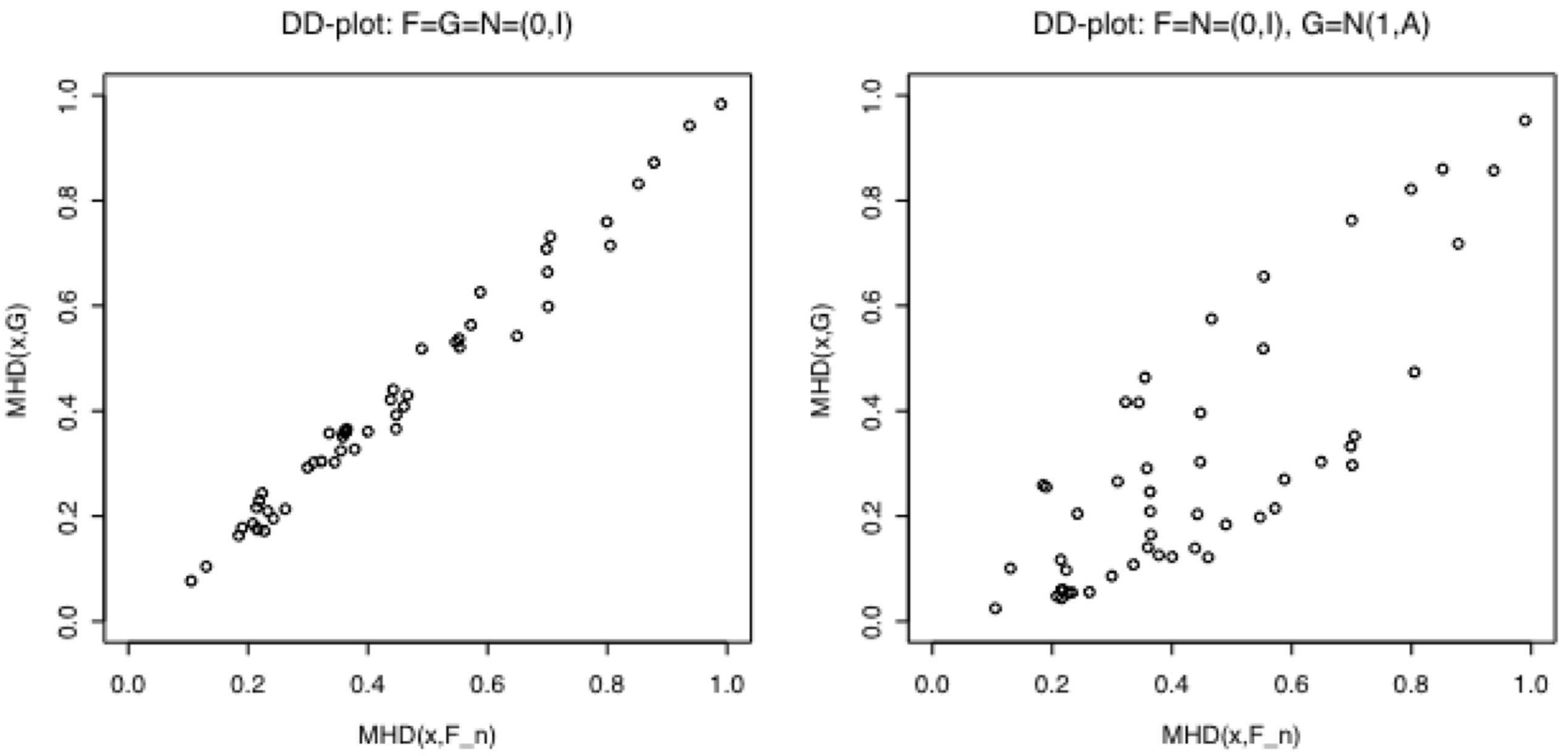

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Female ($n_1 = 24$)			Male ($n_2 = 24$)		
Length	Width	Height	Length	Width	Height
98	81	38	93	74	37
⋮	⋮	⋮	⋮	⋮	⋮
177	132	67	135	106	47

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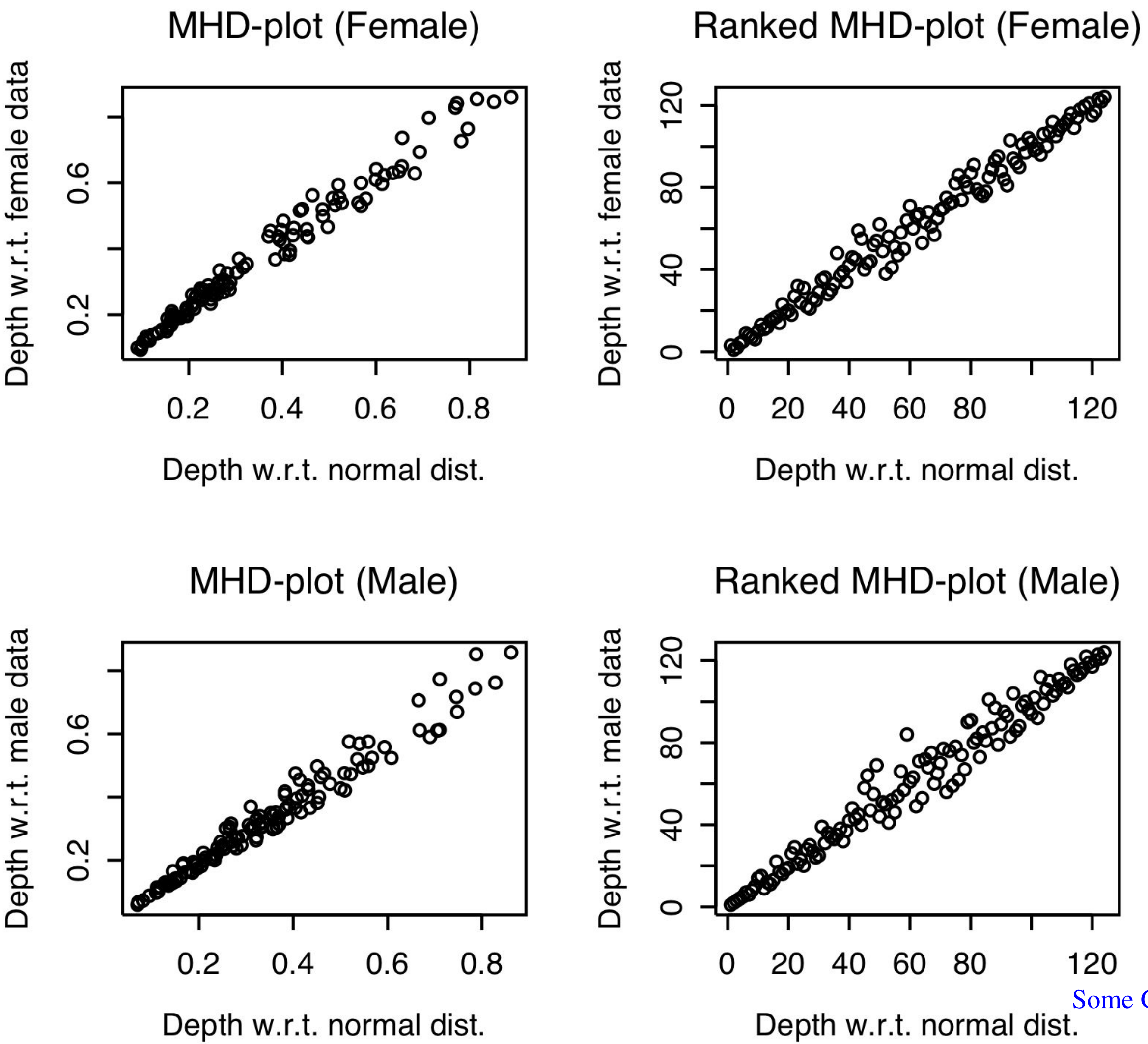

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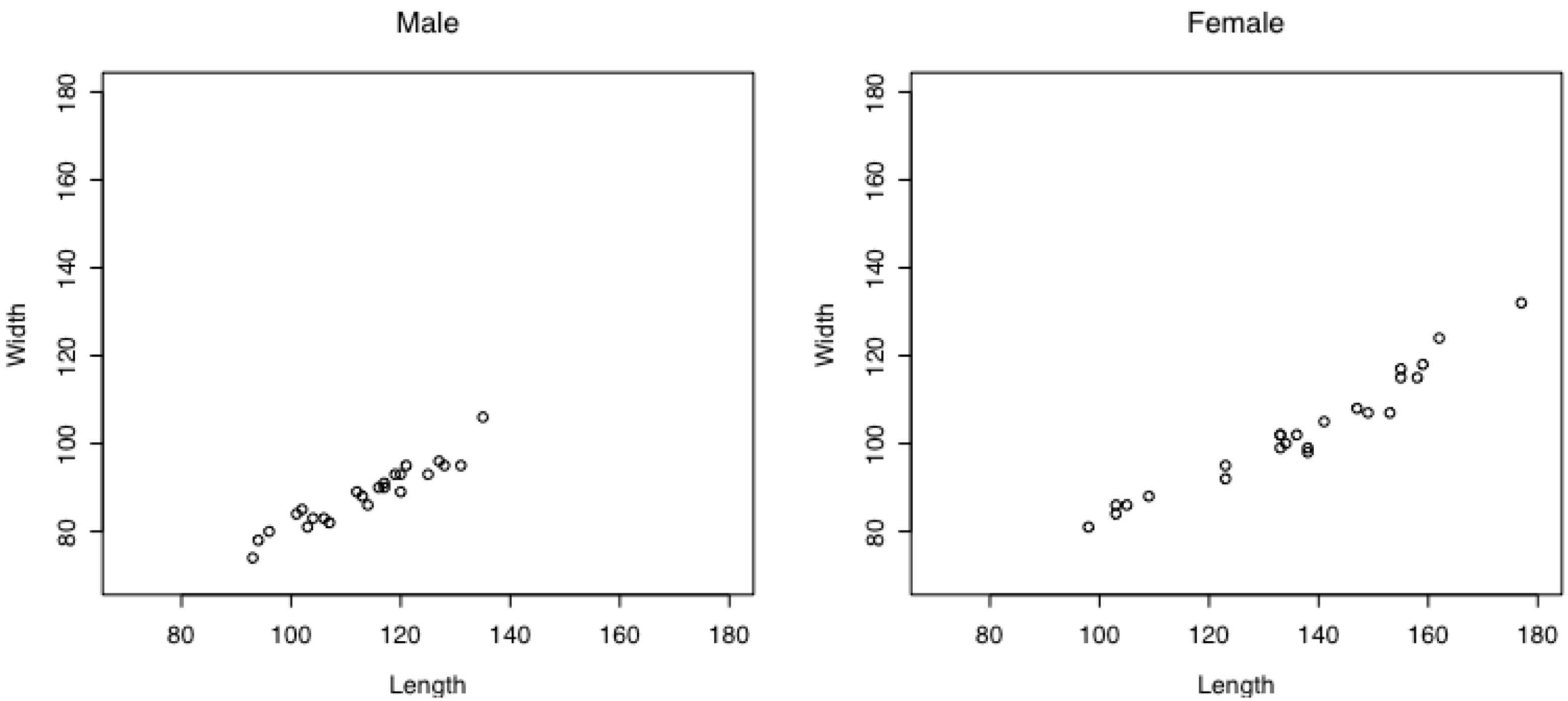

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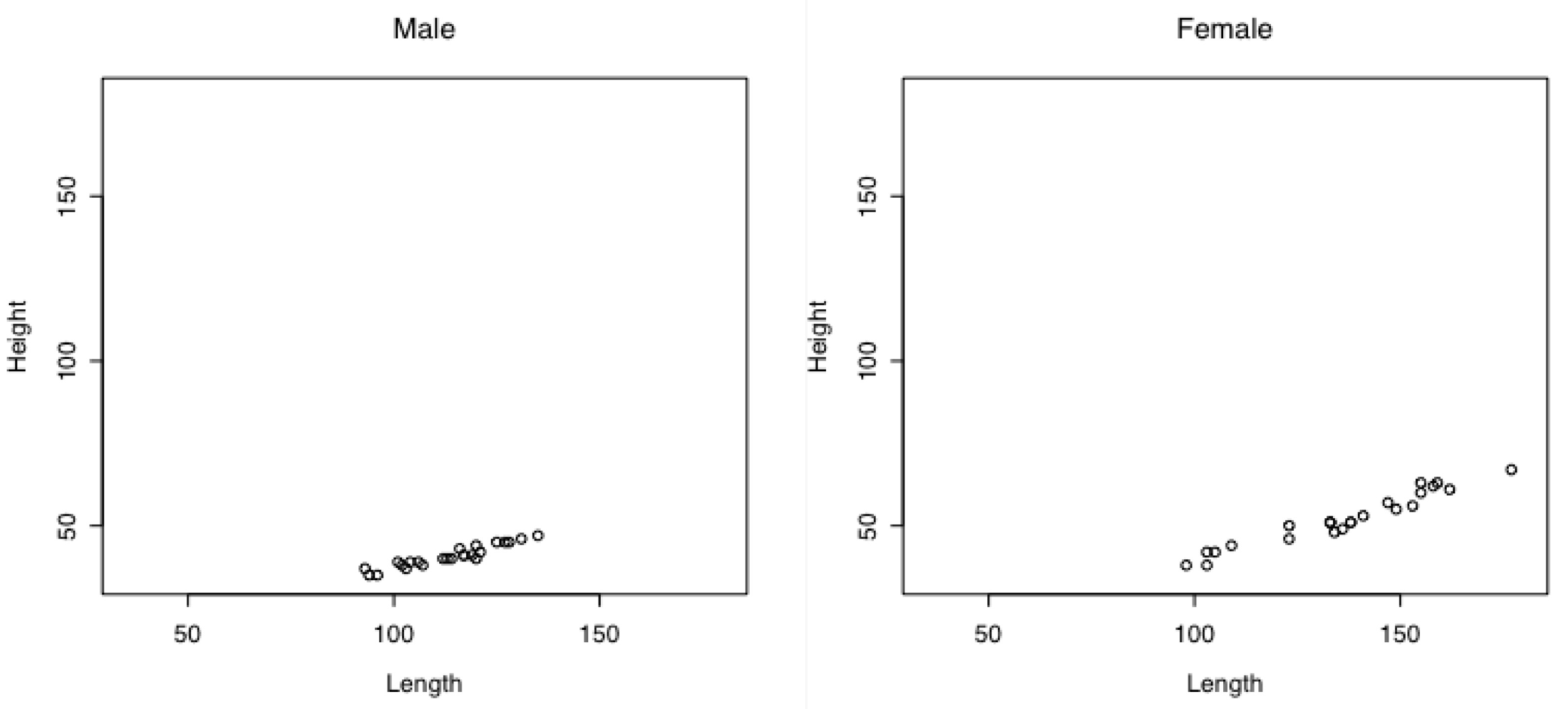

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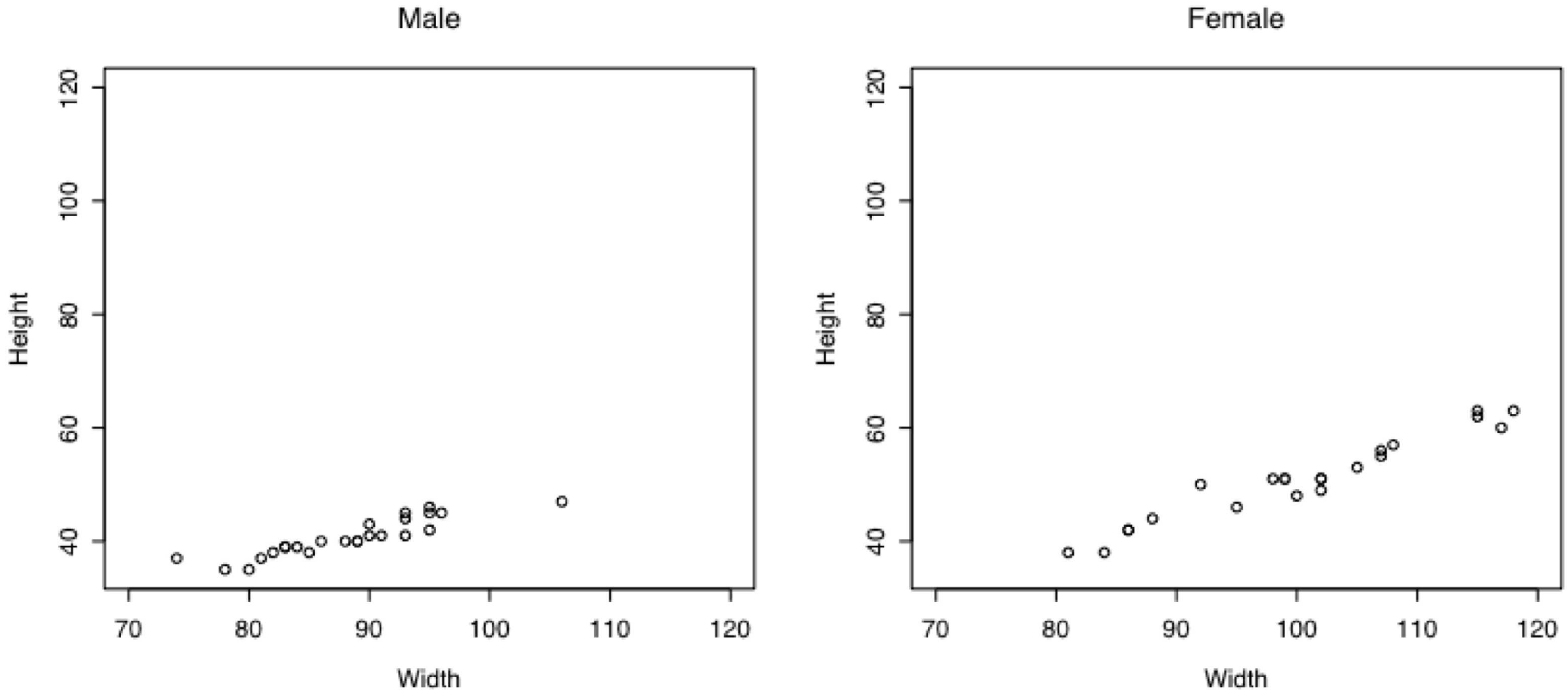

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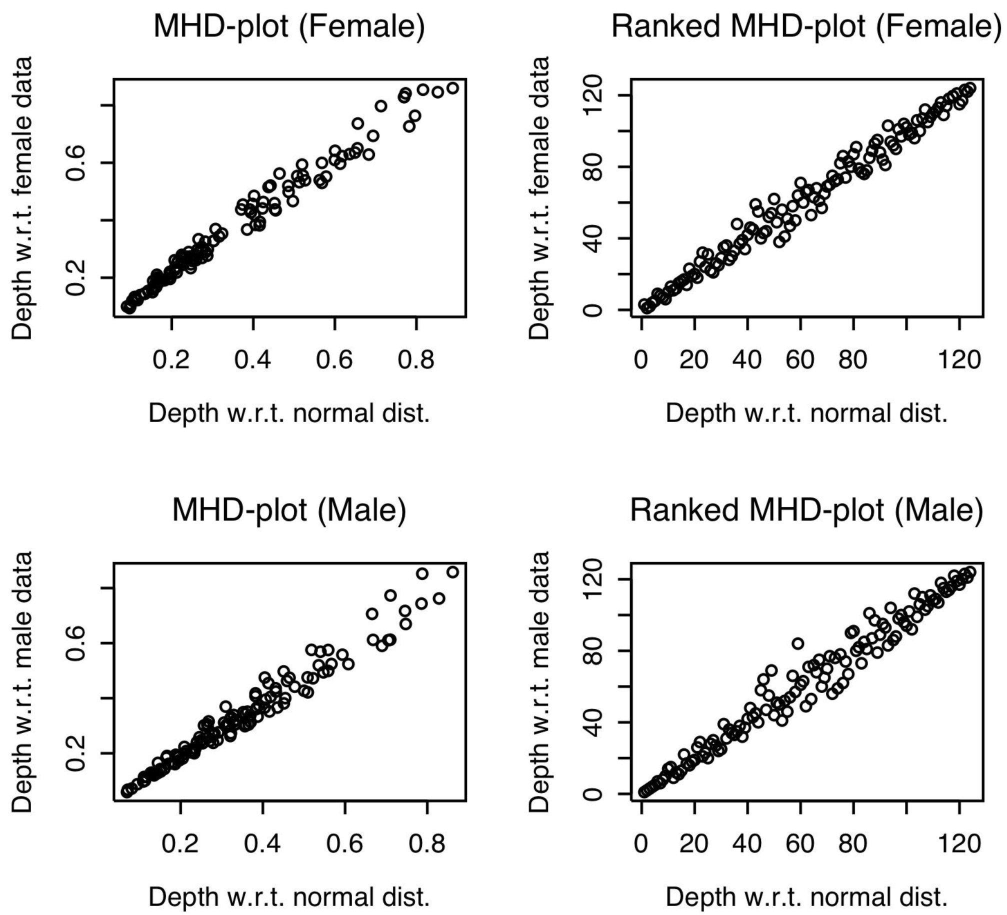

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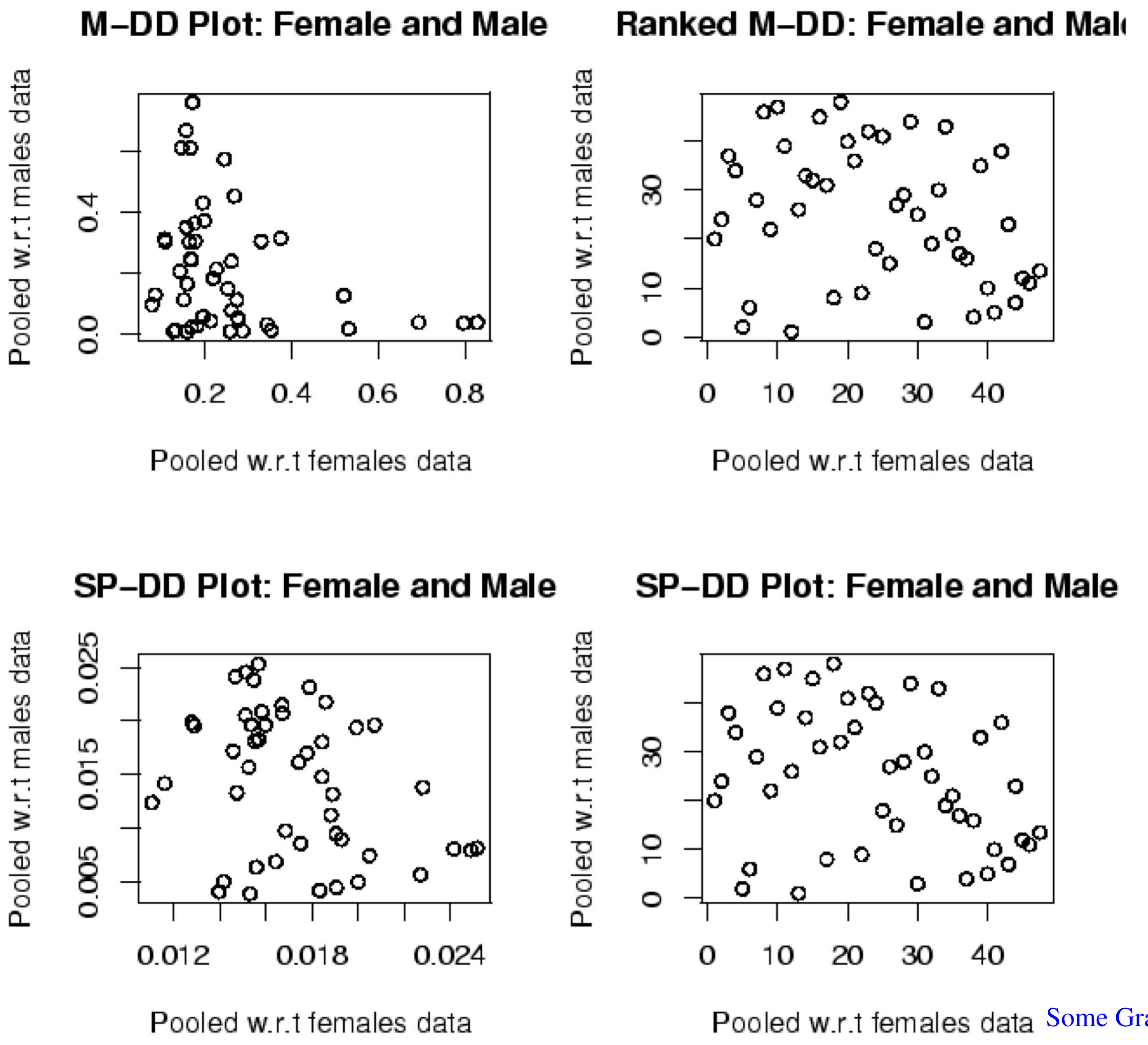

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Proposed Test	Hotelling's T^2
Asymptotic P-Value$\simeq 0$ Exact P-value$\simeq 0$	P-Value$\simeq 0$

Some Other Applications

- **Multivariate descriptive statistics and graphical tools:** Liu et. al (1999), Serfling (2002).
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- **Functional data analysis;** Fraiman and Muniz (2001) and Lopéz-Pintado, and Romo (2003)