# Invariant coordinate selection (ICS): A nonparametric view of independent components analysis (ICA)

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## Abstract

An obvious method for generating measures of location for $p$-dimensional distributions is to simply apply univariate measures of location to each of the coordinates, e.g. the coordinatewise median. A drawback to this approach is that the resulting measure of multivariate location is not affine equivariate. If one could select the coordinates in an invariant manner, however, i.e. select $p$ data dependent linear combinations of the variables which are invariant under nonsingular transformations of the variables, then applying coordinatewise measure of univariate location to the transformed variables and then back-transforming gives an affine equivariant measure of multivariate location. Affine covariant measures for the scatter matrix can also be generated using coordinatewise measures of scale.

To be more specific, let $Y = \{y_1, \ldots, y_n\}$ be a $p$-dimensional data set. Suppose we are able to define a nonsingular matrix $A(Y)$ such that the transformed $p$-dimensional data set $Z = A(Y)Y$ is invariant under nonsingular transformations of $Y$, i.e. $A(Y)Y = A(BY)BY$ for any nonsingular matrix $B$. If we then apply univariate measures of location and scale to each of the components of $Z$ producing $\mu(Z) \in \Re^p$ and $\sigma(Z) \in \Re^p$ respectively, then affine equivariant measures of multivariate location and scatter can be defined by

$$\mu(Y) = A(Y)^{-1}\mu(Z) \quad \text{and} \quad \Sigma(Y) = A(Y)^{-1}D(\sigma^2(Z))(A(Y)')^{-1},$$

where $D(\cdot)$ is a diagonal matrix whose diagonal elements are given by its vector argument.

One method for generating such an invariant transformation is as follows. First compute two different affine covariate estimates of scatter for $Y$, say $V_o$ and $V_1$, and then define $A(Y) = (a_1, \ldots, a_p)$ such that

$$V_o a_j = \gamma_j V_1 a_j \quad \text{for} \quad j = 1, \ldots, p \quad \text{or equivalently,} \quad V_o A(Y) = V_1 A(Y)\Delta,$$

where $\Delta = D(\gamma_1, \ldots, \gamma_p)$. That is, $A(Y)$ are the principal component vectors of $V_o$, relative to the Mahalonobis inner product defined via $V_1$. The transformed variates $Z = A(Y)Y$ can be viewed as *affine invariant principal components*. In a personal communication, Hannu Oja has noted that under certain conditions, the matrix $A(Y)^{-1}$ also represents a solution to the independent component analysis problem.

Beside using the transformed variate $Z$ to generate measures of multivariate location and scatter, these transformed variates can also be used to generate multivariate generalizations of univariate concepts, e.g. affine equivariate quantiles. They can also be used for generating affine invariant nonparametric tests, e.g. an affine invariant sign tests which is asymptotically nonparametric over the class of all symmetric multivariate distributions and not just elliptically symmetric distributions.

Finally, we note that one can produce affine invariant diagnostic plots by plotting the components of $Z$ or by making pairwise scatter plots of the components of $Z$. We give several examples which illustrates the utility of the proposed methods.

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