Models and termination of proof-reduction in the $\lambda\Pi$-calculus modulo theory

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Models and truth values

A model: a set $\mathcal{M}$, a set $\mathcal{B}$, a function (parametrized by valuations) $[\![.]\!]$ mapping terms to elements of $\mathcal{M}$, and propositions to elements of $\mathcal{B}$

E.g.: $[\![A \wedge B]\!]_\phi = [\![A]\!]_\phi \ \tilde{\wedge}\ [\![B]\!]_\phi$

$[\![\forall x\ A]\!]_\phi = \tilde{\forall}\{[\![A]\!]_{\phi+x=a} \mid a \in \mathcal{M}\}$ ($\tilde{\forall}$ from $\mathcal{P}(\mathcal{B})$ to $\mathcal{B}$)

$\mathcal{B} = \{0, 1\}$ but also: a Boolean algebra, a Heyting algebra, a pre-Boolean algebra, a pre-Heyting algebra (pre-order)

Pre-order: distinguish weak equivalence ($[\![A]\!]_\phi \leq [\![B]\!]_\phi$ and $[\![B]\!]_\phi \leq [\![A]\!]_\phi$) from strong $[\![A]\!]_\phi = [\![B]\!]_\phi$

Deduction modulo theory

Theory: axioms + congruence (computational / definitional eq.)

Proofs modulo the congruence

E.g. $(2 \times 2 = 4) \equiv \top$

$$\dfrac{\dfrac{}{\vdash 2 \times 2 = 4}\ \top\text{-intro}}{\vdash \exists x\,(2 \times x = 4)}\ (2)\ \exists\text{-intro}$$

Models and termination in Deduction modulo theory

Proposition A valid if for all ϕ, $[\![A]\!]_\phi \geq \tilde{\top}$

(In particular: $A \Leftrightarrow B$ valid if for all ϕ, $[\![A]\!]_\phi \leq [\![B]\!]_\phi$ and $[\![B]\!]_\phi \leq [\![A]\!]_\phi$)

Congruence $\equiv$ valid if $A \equiv B$ implies for all ϕ, $[\![A]\!]_\phi = [\![B]\!]_\phi$

Note: $\leq$ not used for defining validity of $\equiv$

Proof-reduction does not always terminate $P \equiv (P \Rightarrow P)$

But it does if this theory has a model valued in the pre-Heyting algebra of reducibility candidates (D-Werner 20th century)

The algebra of reducibility candidates

A pre-Heyting algebra but not a Heyting algebra: $(\tilde{\top} \tilde{\Rightarrow} \tilde{\top}) \neq \tilde{\top}$

For termination, congruence matters, not axioms

$\leq$ immaterial, can take $a \leq b$ always: Trivial pre-Heyting algebra

The conditions (e.g. $a \mathrel{\tilde{\wedge}} b \leq a$) always satisfied

A set $\mathcal{B}$ equipped with operations $\tilde{\wedge}$, $\tilde{\Rightarrow}$, $\tilde{\forall}$, ... and no conditions

Super-consistency

Proof-reduction terminates

if $\equiv$ has a model valued in the algebra of reducibility candidates

a fortiori:

if for each trivial pre-Heyting algebra $\mathcal{B}$, $\equiv$ has a $\mathcal{B}$-model

if for each pre-Heyting algebra $\mathcal{B}$, $\equiv$ has a $\mathcal{B}$-model

Model-theoretic sufficient conditions for termination of proof-reduction

From Deduction modulo theory to the $\lambda\Pi$-calculus modulo theory

Deduction modulo theory + algorithmic interpretation of proofs =

$\lambda\Pi$-calculus modulo theory (aka Martin-Löf Logical Framework)

λ-calculus with dependent types + an extended conversion rule

$$\frac{\Gamma \vdash A : s \quad \Gamma \vdash B : s \quad \Gamma \vdash t : A}{\Gamma \vdash t : B} \; A \equiv B$$

Logical Framework: various congruences permit to express proofs in various theories: Arithmetic, Simple type theory, the Calculus of Constructions, functional Pure Type Systems, ...

This talk

What is a model of the $\lambda\Pi$-calculus modulo a congruence $\equiv$?

What is a model valued in a (trivial) pre-Heyting algebra $\mathcal{B}$?

A proof that the existence of such a model implies termination of proof-reduction

An application to a termination proof for proof-reduction in the $\lambda\Pi$-calculus modulo Simple type theory and modulo the Calculus of Constructions

Π-algebras

Adapt notion of (trivial) pre-Heyting algebra to $\lambda\Pi$-calculus

A set $\mathcal{B}$ with two operations $\tilde{T}$ and $\tilde{\Pi}$ and no conditions

$\tilde{T}$ in $\mathcal{B}$ (both for $\top$ and “termination”)

$\tilde{\Pi}$ from $\mathcal{B} \times \mathcal{A}$ to $\mathcal{B}$ ($\mathcal{A}$ subset of $\mathcal{P}(\mathcal{A})$): Π both a binary connective and a quantifier

Double interpretation

Already in Many-sorted predicate logic: a family of domains $(\mathcal{M}_s)_s$ indexed by sorts

Then, $[\![.]\!]$ mapping terms of sort s to elements of $\mathcal{M}_s$ and propositions to elements of $\mathcal{B}$

In the $\lambda\Pi$-calculus, sorts, terms, and propositions are λ-terms:

$(\mathcal{M}_t)_t$ indexed by λ-terms

$[\![.]\!]$ mapping each λ-term t of type A to $[\![t]\!]_\phi$ in $\mathcal{M}_A$

A model valued in $\mathcal{B}$:

on $\mathcal{M}$: $\mathcal{M}_{Kind} = \mathcal{M}_{Type} = \mathcal{B}$

on $[\![.]\!]$: $[\![Kind]\!]_\phi = [\![Type]\!]_\phi = \tilde{T}$

$[\![\Pi x : C\ D]\!]_\phi = \tilde{\Pi}([\![C]\!]_\phi, \{[\![D]\!]_{\phi+x=c} \mid c \in \mathcal{M}_C\})$

Validity of $\equiv$:

if $A \equiv B$ then

$\mathcal{M}_A = \mathcal{M}_B$

and for all ϕ, $[\![A]\!]_\phi = [\![B]\!]_\phi$

Example: a model of the $\lambda\Pi$-calculus modulo simple type theory

$\iota : Type, o : Type,$

$\varepsilon : o \rightarrow Type,$

$\dot{\Rightarrow} : o \rightarrow o \rightarrow o, \dot{\forall}_A : (A \rightarrow o) \rightarrow o$ (for a finite number of A)

Congruence defined by the rewrite rules

$$\varepsilon(\dot{\Rightarrow}\ X\ Y) \longrightarrow \varepsilon(X) \rightarrow \varepsilon(Y)$$

$$\varepsilon(\dot{\forall}_A\ X) \longrightarrow \Pi z : A\ \varepsilon(X\ z)$$

$(\mathcal{M}_t)_t$

$\mathcal{B}$ any Π-algebra and $\{e\}$ any one-element set

- $\mathcal{M}_{Kind} = \mathcal{M}_{Type} = \mathcal{M}_o = \mathcal{B}$
- $\mathcal{M}_\iota = \mathcal{M}_\varepsilon = \mathcal{M}_{\dot{\Rightarrow}} = \mathcal{M}_{\dot{\forall}_A} = \mathcal{M}_x = \{e\}$
- $\mathcal{M}_{\lambda x:C\ t} = \mathcal{M}_t$
- $\mathcal{M}_{(t\ u)} = \mathcal{M}_t$
- $\mathcal{M}_{\Pi x:C\ D}$ set of functions from $\mathcal{M}_C$ to $\mathcal{M}_D$ except if $\mathcal{M}_D = \{e\}$, in which case $\mathcal{M}_{\Pi x:C\ D} = \{e\}$

$[\![.]\!]$

- $[\![Kind]\!]_\phi = [\![Type]\!]_\phi = [\![\iota]\!]_\phi = [\![o]\!]_\phi = \tilde{T}$
- $[\![\lambda x : C\ t]\!]_\phi$ function ...
- $[\![\Pi x : C\ D]\!]_\phi = \tilde{\Pi}([\![C]\!]_\phi, \{[\![D]\!]_{\phi, x=c} \mid c \in \mathcal{M}_C\})$
- $[\![\varepsilon]\!]_\phi$ is the identity on $\mathcal{B}$
- ...

Also (but more complicated): a model of the $\lambda\Pi$-calculus modulo the Calculus of Constructions

Termination of proof-reduction

Theorem: if a $\equiv$ has a model valued in all (trivial) pre-Heyting algebras then proof-reduction modulo $\equiv$ terminates

Business as usual

A model valued in the algebra of reducibility candidates

$[\![A]\!]_\phi$ set of terms

if $t : A$ then $t \in [\![A]\!]$ hence t terminates

Conclusion

Usual "Tarskian" notion of model valued in an algebra $\mathcal{B}$ extends to type theory: no conceptual difficulties (but devil in the details)

A purely model-theoretic sufficient condition for termination of proof-reduction

Applies to Simple type theory and the Calculus of Constructions

Future work: non-trivial pre-orders $\leq$ to prove independence results without the detour to termination of proof-reduction