Logics extended with embedding-closed quantifiers

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- A quantifier Q corresponds to some property P_Q of τ-structures for a given vocabulary τ
- By adding a quantifier Q to a logic $\mathcal{L}$ we get the smallest extension $\mathcal{L}$ that can express property P_Q

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Example (Equicardinality quantifier I)

$\mathfrak{A} \models Ixy(\varphi(x), \psi(y))$ if and only if φ and ψ define sets of the same cardinality

Embedding-closed quantifiers

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- Suppose $(\mathfrak{A}, \overline{a}) \vDash Q(\overline{x}_\alpha \vartheta_\alpha)_{\alpha<\kappa}$ and $f : \mathfrak{A} \to \mathfrak{B}$ is an embedding

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- Thus $(\mathfrak{B}, f\overline{a}) \models Q(\overline{x}_\alpha \vartheta_\alpha)_{\alpha<\kappa}$

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Lemma

A τ-structure $\mathfrak{A}$ has quantifier elimination for $\mathcal{L}_{\infty\omega}(\mathcal{Q}_{\mathrm{emb}})$ if and only if it is quasi-homogeneous.

Preservation of formulas in chains of quasi-homogeneous structures

Theorem

Let τ be a finite relational vocabulary, $\mathcal{Q}$ a finite set of embedding-closed quantifiers of finite width and $(\mathfrak{A}_i)_{i<\omega}$ a chain of quasi-homogeneous τ-structures. For each $m < \omega$, there is a natural number N_m such that for every formula $\varphi \in \mathcal{L}^m_{\infty\omega}(\mathcal{Q})[\tau]$ there is a quantifier-free τ-formula ϑ_φ such that

$$\mathfrak{A}_i \models \forall \overline{x}(\varphi \leftrightarrow \vartheta_\varphi)$$

for all $i \geq N_m$.

Preservation of formulas in chains of quasi-homogeneous structures

$$\mathfrak{A}_0 \leq \mathfrak{A}_1 \leq \cdots \leq \underbrace{\mathfrak{A}_{N_m} \leq \cdots}_{\mathfrak{A}_i \models \forall \overline{x}(\varphi \leftrightarrow \vartheta_\varphi)}$$

Proof.

- Let $\varphi_0, \ldots, \varphi_l$ be an enumeration of $\mathcal{L}^m_{\infty\omega}(\mathcal{Q})$-formulas of the form $Q(\overline{x}_i\psi_i)_{i<\omega}$ with all ψ_i quantifier-free, and suppose φ is one of these formulas

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- For each $i < \omega$, let $T_i = \{t : t$ is an atomic type and $(\mathfrak{A}_i, \overline{a}) \vDash \varphi \wedge t$ for some $\overline{a}\}$.

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- Since both φ and $\bigvee T_i$ are preserved in embeddings, we have $T_i \subseteq T_j$ always when $i \leq j$ so T_i:s reach their maximum at some $k_\varphi < \omega$

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- Thus we can set $N_m = \max\{k_{\varphi_i} : i \leq l\}$

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This proof can be generalized to formulas of the logic $\mathcal{L}_{\infty\omega}(\mathcal{Q}_{\mathrm{emb}})$ as well.

Corollary

If $\mathfrak{A}$ and $\mathfrak{B}$ are quasi-homogeneous bi-embeddable structures then $\mathfrak{A} \equiv_{\mathrm{emb}} \mathfrak{B}$.

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Example

The following properties are not definable in $\mathcal{L}_{\infty\omega}(\mathcal{Q}_{\text{emb}})$:

- Equicardinality of unary predicates
- Completenes of an order
- Cofinality of an order

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- Duplicator wins if the game reaches position with $\beta = 0$.

Embedding game

Theorem

Let τ *be vocabulary and* $\mathfrak{A}$, $\mathfrak{B}$ τ*-structures. For all ordinals* $\gamma < \omega_1$ *we have* $\mathfrak{A} \simeq^{\gamma}_{\mathrm{emb}} \mathfrak{B}$ *if and only if* $\mathfrak{A} \equiv^{\gamma}_{\mathrm{emb}} \mathfrak{B}$.

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Corollary

Let τ be vocabulary and $\mathfrak{A}$, $\mathfrak{B}$ τ-structures. Then we have $\mathfrak{A} \simeq_{\text{emb}} \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_{\text{emb}} \mathfrak{B}$.

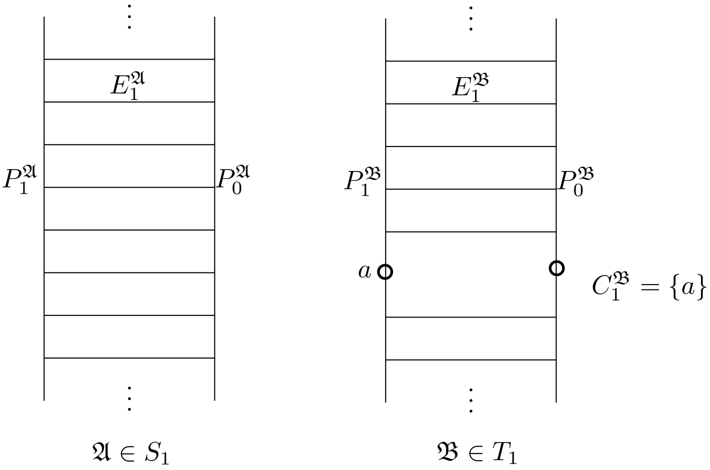
Embedding game

We use embedding game to prove the following:

Theorem

For each $n < \omega$, there is a first-order sentence φ_n of quantifier rank n that is not expressible by any $\mathcal{L}_{\infty\omega}(\mathcal{Q}_{\mathrm{emb}})$-sentence of quantifier rank less than n.

Embedding game



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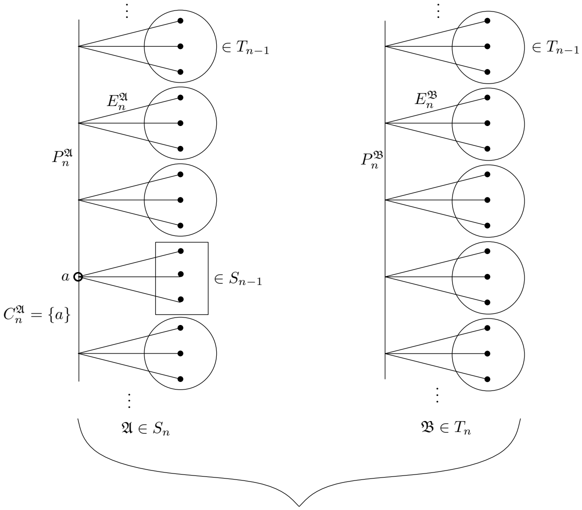

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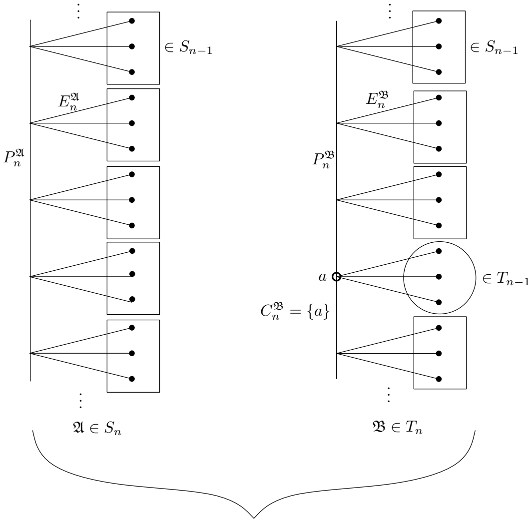

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Thank you!