Hierarchies in inclusion logic

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- We will consider expressivity hierarchies within inclusion logic, written $\mathrm{FO}(\subseteq)$, under two different semantics:
  - lax team semantics,
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- These hierarchies arise from the syntactical fragments:
  - $\mathrm{FO}(\subseteq)(k\text{-inc})$,
  - $\mathrm{FO}(\subseteq)(k\forall)$,

  defined by restricting the arity of inclusion atom or the number of universal quantifiers, respectively.

Introduction I

Inclusion logic is one part of the family of logics that extend first-order logic with different dependency notions. This family of logics arises from dependence logic (Väänänen 2007) which extends first-order logic with dependence atoms

$$=(x_1, \dots, x_n)$$

expressing that the values of x_n depend functionally on the values of $x_1, \dots, x_{n-1}$.

Introduction II

Inclusion logic, instead, extends first-order logic with inclusion atoms

$$x_1 \ldots x_n \subseteq y_1 \ldots y_n$$

which express that the set of values of $(x_1, \ldots, x_n)$ is included in the set of the values of $(y_1, \ldots, y_n)$.

Syntax of $\mathrm{FO}(\subseteq)$

The syntax of $\mathrm{FO}(\subseteq)$ is given by the following grammars:

$$\phi ::= x_1 \dots x_n \subseteq y_1 \dots y_n \mid t_1 = t_2 \mid \neg t_1 = t_2 \mid R(\vec{t}) \mid \neg R(\vec{t}) \mid (\phi \wedge \psi) \mid (\phi \vee \psi) \mid \forall x \phi \mid \exists x \phi.$$

Team semantics of $\mathrm{FO}(\subseteq)$

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Let $\mathcal{M}$ be a model with domain M. Then an *assignment* over M is a finite function that maps variables to elements of M. A *team* X of M with the domain $\mathrm{Dom}(X) = \{x_1, \ldots, x_n\}$ is a set of assignments from $\mathrm{Dom}(X)$ into M.

Team semantics of $\mathrm{FO}(\subseteq)$ (cases where strict = lax)

We define two different semantics for inclusion logic, the so-called strict and lax team semantics. For FO-literals, $\subseteq$-atoms, $\wedge$ and $\forall$, the (lax and strict) semantic rules are the following. Let $\mathcal{M}$ be a model with domain M and X a team of M. Then we let:

- FO-lit: For all first-order literals α, $\mathcal{M} \models_X \alpha$ if and only if, for all $s \in X$, $\mathcal{M} \models_s \alpha$ in the usual Tarski semantics sense;
- $\subseteq$: $\mathcal{M} \models_X x_1 \ldots x_n \subseteq y_1 \ldots y_n$ if and only if for all $s \in X$ there exists an $s' \in X$ such that $s(x_i) = s'(y_i)$, for $i = 1, \ldots, n$;
- $\wedge$: For all ψ and θ, $\mathcal{M} \models_X \psi \wedge \theta$ if and only if $\mathcal{M} \models_X \psi$ and $\mathcal{M} \models_X \theta$;
- $\forall$: For all ψ and all variables v, $\mathcal{M} \models_X \forall v \psi$ if and only if $\mathcal{M} \models_{X[M/v]} \psi$, where $X[M/v] = \{s[m/v] : s \in X, m \in M\}$.

Team semantics of $\mathrm{FO}(\subseteq)$ (cases where strict $\neq$ lax)

For $\vee$ and $\exists$, the strict and lax semantics are defined differently. The semantic rules for disjunction are as follows:

lax-$\vee$: For all ψ and θ, $\mathcal{M} \models_X \psi \vee \theta$ if and only if there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $\mathcal{M} \models_Y \psi$ and $\mathcal{M} \models_Z \theta$;

strict-$\vee$: For all ψ and θ, $\mathcal{M} \models_X \psi \vee \theta$ if and only if there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \cap Z = \emptyset$, $\mathcal{M} \models_Y \psi$ and $\mathcal{M} \models_Z \theta$.

Team semantics of $\mathrm{FO}(\subseteq)$ (cases where strict $\neq$ lax) cont.

The semantic rules for existential quantification are as follows:

lax-$\exists$: For all ψ and all variables v, $\mathcal{M} \models_X \exists v \psi$ if and only if there exists a function $H : X \to \mathcal{P}(\mathcal{M}) \setminus \{\emptyset\}$ such that $\mathcal{M} \models_{X[H/v]} \psi$ where $X[H/v] := \{s[m/v] : s \in X, m \in H(s)\}$;

strict-$\exists$: For all ψ and all variables v, $\mathcal{M} \models_X \exists v \psi$ if and only if there exists a function $H : X \to M$ such that $\mathcal{M} \models_{X[H/v]} \psi$ where $X[H/v] := \{s[m/v] : s \in X, m = H(s)\}$.

Team semantics of $\mathrm{FO}(\subseteq)$ (cases where strict $\neq$ lax) cont.

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From now on, let us write $\models^{\mathrm{L}}$ and $\models^{\mathrm{S}}$ for the lax and strict team semantics, respectively.

Properties I

First-order logic is embedded in $\mathrm{FO}(\subseteq)$ in the following sense. Here $\models$ refers to the Tarskian semantics.

Theorem (Flatness)

For a model $\mathcal{M}$, a first-order formula ϕ and a team X, the following are equivalent:

- $\mathcal{M} \models^{\mathrm{L}}_X \phi$,
- $\mathcal{M} \models^{\mathrm{S}}_X \phi$,
- $\mathcal{M} \models_s \phi$ *for all* $s \in X$.

Properties II

Theorem (Locality)

Let $\mathcal{M}$ be a model, X be a team, $\phi \in \mathrm{FO}(\subseteq)$ and V a set of variables such that $Fr(\phi) \subseteq V \subseteq Dom(X)$. Then

$$\mathcal{M} \models^{\mathrm{L}}_{X} \phi \Leftrightarrow \mathcal{M} \models^{\mathrm{L}}_{X \upharpoonright V} \phi.$$

For $\models^{\mathrm{S}}$, this principle fails as illustrated in the following example.

Properties cont.

Example

Let $\mathcal{M} = \{0, 1, 2\}$ and let X be as in the picture.

	x	y	z	v
s_0	0	1	2	0
s_1	1	0	1	0
s_2	1	0	1	1
s_3	2	1	0	0

Then $\mathcal{M} \models^{\mathrm{S}}_{X} x \subseteq y \vee z \subseteq y$, since we can choose $Y := \{s_0, s_1\}$ and $Z := \{s_2, s_3\}$.

Properties cont.

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Then $\mathcal{M} \models^{\mathrm{S}}_X x \subseteq y \vee z \subseteq y$, since we can choose $Y := \{s_0, s_1\}$ and $Z := \{s_2, s_3\}$. However, taking $X' := X \upharpoonright \{x, y, z\}$, we obtain that $\mathcal{M} \not\models^{\mathrm{S}}_X x \subseteq y \vee z \subseteq y$, since X' is the below team.

	x	y	z
s_0	0	1	2
s_1	1	0	1
s_3	2	1	0

Expressive power

Under the lax team semantics the following holds.

Theorem (Galliani, Hella 2013)

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Every inclusion logic sentence is equivalent to a greatest fixed point logic sentence, and vice versa.

Under the strict team semantics the following holds.

Theorem (Galliani, H., Kontinen 2013)

Every inclusion logic sentence is equivalent to a existential second-order logic sentence, and vice versa.

Expressive power cont.

Now, using well-known results of descriptive complexity theory, we obtain the following corollary.

Corollary

- *With* $\models^{\mathrm{L}}$*: a class* $\mathcal{C}$ *of finite linearly ordered models is definable in* $\mathrm{FO}(\subseteq)$ *if and only if it can be recognized in* PTIME*.*
- *With* $\models^{\mathrm{S}}$*: a class* $\mathcal{C}$ *of finite models is definable in* $\mathrm{FO}(\subseteq)$ *if and only if it can be recognized in* NP*.*

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Recall the semantic rules for the lax and the strict versions.

Expressive power cont.

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Recall the semantic rules for the lax and the strict versions. A strange observation:

- $\mathrm{FO}(\subseteq)$ with *non-deterministic* existential quantification captures *deterministic* polynomial time.
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Syntactical fragments in $\mathrm{FO}(\subseteq)$

Next we define two syntactical fragments of inclusion logic.

Definition

- $\mathrm{FO}(\subseteq)(k\text{-inc})$, is the class of formulae $\phi \in \mathrm{FO}(\subseteq)$ where ϕ may contain at most k-ary inclusion atoms (i.e. atoms of the form $x_1 \dots x_n \subseteq y_1 \dots y_n$ where $n \leq k$).
- $\mathrm{FO}(\subseteq)(k\forall)$ is the class of formulae $\phi \in \mathrm{FO}(\subseteq)$ where ϕ may contain at most k occurrences of the quantifier $\forall$.

First we will consider $\mathrm{FO}(\subseteq)(k\forall)$-fragments with both semantics.

$\forall$-hierarchies (with lax)

For logics $\mathcal{L}$ and $\mathcal{L}'$, we write $\mathcal{L} \leq \mathcal{L}'$, if for every signature τ, every $\mathcal{L}[\tau]$-sentence is logically equivalent to some $\mathcal{L}'[\tau]$-sentence. Equality and inequality relations are obtained from $\leq$ naturally.

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Theorem (H.)

$\mathrm{FO}(\subseteq)(1\forall) = \mathrm{FO}(\subseteq)$.

Proof.

Sketch. The result holds already at the level of formulae, so let $\phi \in \mathrm{FO}(\subseteq)$ be a formula. W.l.o.g. we may assume that ϕ is of the form $Q^1 x_1 \dots Q^n x_n \theta$ where θ is quantifier-free. We let

$$\phi' := \exists x_1 \dots \exists x_n \forall y(\bigwedge_{\substack{1 \leq i \leq n \\ Q^i = \forall}} \vec{z} x_1 \dots x_{i-1} y \subseteq \vec{z} x_1 \dots x_{i-1} x_i \wedge \theta)$$

where $\vec{z}$ lists $\mathrm{Fr}(\phi)$. Clearly $\phi' \in \mathrm{FO}(\subseteq)(1\forall)$. Also we obtain that $\phi \equiv \phi'$. □

$\forall$-hierarchies (with strict)

Recall that under the strict semantics, inclusion logic is as expressive as existential second-order logic (ESO). Hence, we will try to relate the universal fragments of $\mathrm{FO}(\subseteq)$ to the corresponding fragments of ESO, defined as follows:

Definition

$\mathrm{ESO}_f(k\forall)$ is the class of skolem normal form ESO-sentences

$$\exists f_1, \ldots, f_n \forall x_1 \ldots \forall x_m \psi,$$

where $m \leq k$.

$\forall$-hierarchies (with strict) cont.

Under the assumption that in $\mathrm{FO}(\subseteq)(k\forall)$ each variable is quantified at most once (no reusing of variables), we actually find out that the universal fragments of $\mathrm{FO}(\subseteq)$ and ESO are equivalent.

Theorem (H., Kontinen 2014)

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∀-hierarchies (with strict) cont.

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Theorem (H., Kontinen 2014)

$\mathrm{FO}(\subseteq)(k\forall) = \mathrm{ESO}_f(k\forall)$.

Therefore, we obtain the following hierarchy:

Corollary

$\mathrm{FO}(\subseteq)(k\forall) < \mathrm{FO}(\subseteq)((k+1)\forall)$.

Proof.

Follows from the above theorem, since $\mathrm{ESO}_f(k\forall)$-fragments can be related to the strict degree hierarchy within non-deterministic polynomial time (of random access machines) (Cook 1972 and Grandjean, Olive 2003). □

Hierarchies in $\mathrm{FO}(\subseteq)$ thus far

For an increasing (with respect to $\leq$) sequence of logics $(\mathcal{L}_k)_{k\in\mathbb{N}}$, we say that the $\mathcal{L}_k$-hierarchy collapses at level m if $\mathcal{L}_m = \bigcup_{k\in\mathbb{N}} \mathcal{L}_k$. An $\mathcal{L}_k$-hierarchy is called strict if $\mathcal{L}_k < \mathcal{L}_{k+1}$ for all $k \in \mathbb{N}$.

Hierarchies in $\mathrm{FO}(\subseteq)$ thus far

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	$\forall$-hierarchy	arity hierarchy
$\models^{\mathrm{L}}$	collapse at 1	?
$\models^{\mathrm{S}}$	strict	?

Arity hierarchies (with lax)

Theorem (H. 2014)

$\mathrm{FO}(\subseteq)(k\text{-inc}) < \mathrm{FO}(\subseteq)(k+1\text{-inc})$.

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Idea of the proof. Analogous arity hierarchies for various fixed point logics $(\mathrm{LFP}, \mathrm{IFP}, \mathrm{PFP}, \mathrm{TC})$ were proved in the early 90s (Grohe). Applying this work, one can show that there exists a graph property that separates $\mathrm{FO}(\subseteq)(k\text{-inc})$ and $\mathrm{FO}(\subseteq)(k+1\text{-inc})$. Namely, we let $\phi(x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1})$ be a first-order formula expressing that the variables $x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1}$ form a clique in a graph. Then we show that

$$\neg[\mathrm{TC}_{\vec{x},\vec{y}}\phi](\vec{a}, \vec{b})$$

is expressible in $\mathrm{FO}(\subseteq)(k+1\text{-inc})$ but not in $\mathrm{FO}(\subseteq)(k\text{-inc})$.

Hierarchies in $\mathrm{FO}(\subseteq)$

	$\forall$-hierarchy	arity hierarchy
$\models^{\mathrm{L}}$	collapse at 1	strict
$\models^{\mathrm{S}}$	strict	?

Thanks!

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