Finding independence notions in metric structures

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Why independence?

- aim: classification of structures by sets of invariants
- well behaved independence notions give dimensions
- generalising what forking does in first order theories

Challenges of metric structures

- not FO-axiomatisable
- “look better if blurred”
  - measure size wrt densities
  - allowing for small changes (*perturbations*) make model classes even better behaved (we come down in the stability hierarchy)

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- use syntax-free approach (metric AEC)
- work with respect to perturbations
- look at "classical" independence notion from homogeneous model theory
- approximate this notion

Galois-types

When looking at submodels of a strongly homogeneous monster $\mathfrak{M}$, Galois-types are orbits:

Definition

$$\mathrm{t}^{g}(a/A) = \mathrm{t}^{g}(b/A)$$

if and only if

$$\text{there is } f \in \mathrm{Aut}(\mathfrak{M}/A) \text{ s.t. } f(a) = b.$$

Metrics on the type space

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if there are ε-*automorphisms* f and g of the monster model such that $d(f(a), b) \leq \varepsilon$ and $d(g(b), a) \leq \varepsilon$.

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Extend to types over parameters by:

$$\mathbf{d}^p(\mathrm{t}^g(a/A), \mathrm{t}^g(b/A)) = \sup\{\mathbf{d}^p(\mathrm{t}^g(ac/\emptyset), \mathrm{t}^g(bc/\emptyset)) : c \in A \text{ finite}\}.$$

With suitable assumptions on the ε-isomorphisms this defines a metrisable (diagonal) uniformity.

A note on stability

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metric	$\lvert A\rvert \leq \lambda \Rightarrow \mathrm{dens}(S(A)) \leq \lambda$
with perturbations	$\lvert A\rvert \leq \lambda \Rightarrow \mathrm{dens}(S(A)) \leq \lambda$ with respect to a perturbation metric $\mathbf{d}_{\mathfrak{p}}$

Independence from homogeneous model theory

Fact (Hyttinen–Shelah [HS00])

In a stable homogeneous class there is an independence notion, defined via strong splitting, that works well over saturated enough models.

Assumptions used

We work with a metric abstract elementary class with perturbations $(\mathbb{K}, \preccurlyeq, \mathbb{F}_\varepsilon)_{\varepsilon \geq 0}$ that

- has arbitrarily large models
- satisfies the joint embedding property
- satisfies a form of amalgamation wrt $\mathbb{F}$:
- is homogeneous

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- is homogeneous
- has the perturbation property
- has complete type spaces
- is $\mathbf{d}^p$-superstable (i.e. stable wrt $\mathbf{d}^p$ from some λ onwards)
- is weakly simple

Measuring independence, background

- We have an independence notion and a notion of distance between types.
- But independence is not really a property of Galois types but of Lascar strong types. So we develop a measure of distance for Lascar types.

Lascar strong types and Lascar types

Definition

Two tuples a and b have the same *Lascar strong type over A*,

$$Lstp(a/A) = Lstp(b/A)$$

if $E(a, b)$ holds for any A-invariant equivalence relation with a bounded number of equivalence classes.

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Definition (Hyttinen–Kesälä)

Two tuples a and b have the same *Lascar type over A*,

$$Lstp^w(a/A) = Lstp^w(b/A)$$

if for all finite $B \subseteq A$, $Lstp(a/B) = Lstp(b/B)$.

By homogeneity $Lstp^w(a/A) = Lstp^w(b/A)$ implies $\mathrm{t}^g(a/A) = \mathrm{t}^g(b/A)$.

Measuring independence

We define d_a^p, a distance-like relation on the space of Lascar types, that defines a metrisable topology.

Definition

1. For a finite set A we define

$$d_a^p(Lstp(a/A), Lstp(b/A)) = \sup\{d^p(t^g(a/B), t^g(b/B)) : A \subseteq B \textit{ finite}, B \downarrow_A ab\}.$$

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2. For any set B we then define

$$d_a^p(Lstp^w(a/B), Lstp^w(b/B)) = \sup\{d_a^p(Lstp(a/A), Lstp(b/A)) : A \subseteq B,\ A \textit{ finite}\}.$$

Measuring independence, ε-freeness

Our measure of independence is given by ε-freeness:

Definition

For $\varepsilon > 0$, we write

$$a \downarrow_A^\varepsilon B$$

if for all finite $C \subseteq A$, there is finite $C \subseteq D \subseteq A$ and b such that

$$Lstp(b/D) = Lstp(a/D),\ b \downarrow_D AB \text{ and } d_a^p(Lstp^w(b/AB), Lstp^w(a/AB)) \leq \varepsilon.$$

By $a \downarrow_A^0 B$ we mean that $a \downarrow_A^\varepsilon B$ holds for all $\varepsilon > 0$.

Results

Theorem

Under the given assumptions

1. $a \downarrow^0_A B$ *if and only if* $a \downarrow_A B$.
2. *For all* A *and* a, *there is countable* $B \subseteq A$ *such that* $a \downarrow_B A$.

Note: By [HS00], in a stable homogeneous classes there is $\kappa(\mathbb{K}) < \beth_{(2^{LS(\mathbb{K})})^+}$ such that for all a and $\lambda(\mathbb{K})$-saturated $\mathcal{A}$ there is $A \subseteq \mathcal{A}$ of power $< \kappa(\mathbb{K})$ such that $a \downarrow_A \mathcal{A}$.
But even in the first-order case $\kappa(\mathbb{K})$ cannot be chosen to be smaller than $LS(\mathbb{K})^+$.

Results

Theorem

Under the given conditions, $\mathbb{K}$ is simple, in the given setting superstability and weak simplicity imply simplicity.

Note that weak simplicity does not in general imply simplicity. There is an example of a class that is homogeneous, stable and weakly simple but not simple.

Results

Theorem

If the class $\mathbb{K}$ is stable and weakly simple then T.F.A.E.

(i) $\mathbb{K}$ *is* d^p*-superstable.*

(ii) *For no* $\varepsilon > 0$ *is there an infinite* $\downarrow^{\varepsilon}$*-forking sequence.*

(iii) *For all* a, A *and* $\varepsilon > 0$, *there is a finite* $B \subseteq A$ *such that* $a \downarrow^{\varepsilon}_{B} A$.

Example: p-adic integers

Definition

The *p-adic norm*, $\|\|_p$ is defined by

$$\|a\|_p = p^{-max\{k : p^k | a\}}.$$

The *p-adic integers*, $\mathbb{Z}_p$, is the completion of the integers in the p-adic topology.

The p-adic integers for an *ultrametric space*, i.e. a metric space where the triangle inequality is strengthened to

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

p-adics as a MAEC

Let $\mathbb{K}_p$ be the class of completions of direct sums of copies of the p–adic integers, $\overline{\mathbb{Z}_p^{(\kappa)}}$, with κ any cardinal.

Let $A \preccurlyeq B$ if A is a closed pure subgroup of B.

Use the infimum distance metric.

This gives a class satisfying the assumptions.

Independence in $\mathbb{K}_p$

Lemma

For elements $a \in M \in \mathbb{K}_p$, $A, B \subset M$, $a \downarrow_A B$ *if and only if* $d(a, \langle A \rangle_P) = d(a, \langle AB \rangle_P)$, *where* $\langle A \rangle_P$ *is the pure subgroup in* M *generated by* A.

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Lemma

For elements a in models of $\mathbb{K}_p$

$$\text{if } b \not\downarrow_A B \text{ then } b \not\downarrow_A^{\varepsilon} B \text{ for every } \varepsilon < d(b, \langle A \rangle_P).$$

Finding a pregeometry in $\mathfrak{M}^{eq}$

Let D be the set of all realisations of some unbounded $p = Lstp^w(a/A)$.

Let E be an A-invariant equivalence relation. Denote $a^* = a/E$. We define in D/E a closure operator by

$$a^* \in cl(b_1^*, \ldots, b_n^*)$$

if for all $a' \in a^*$ and $b_i' \in b_i^*$, $i = 1, \ldots, n$,

$$a' \not\downarrow_A b_1' \ldots b_n'.$$

For an arbitrary $B^* \subseteq D/E$ we define $a^* \in cl(B^*)$ if $a^* \in cl(B_0^*)$ for some finite $B_0^* \subseteq B^*$.

Finding a pregeometry in $\mathfrak{M}^{eq}$

Theorem

Assume A is finite or the perturbation system is almost summable. If there is $\varepsilon > 0$ such that for all $b \in D$ and $B \subseteq D$ the following are equivalent:

1. $b \not\downarrow_A B$
2. $b \not\downarrow_A^{>\varepsilon} B$
3. *for all $c \in D$ there exists $b' \in b^*$ such that $c \downarrow_{AB}^{>\varepsilon} b'$.*

then $(D/E, cl)$ is a pregeometry.

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then $(D/E, cl)$ is a pregeometry.

Theorem

In $\mathbb{K}_p$ the condition above holds for a type of a single element when E is chosen to be the relation $\not\downarrow_A$.

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