Simple homogeneous structures

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Introduction

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I will present some results in the intersection of these areas, i.e. we consider structures that are **both simple and homogeneous**. References (containing more references) follow at the end.

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1. $\mathcal{M}$ has elimination of quantifiers.
2. Every isomorphism between finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.
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Via the Engeler, Ryll-Nardzewski, Svenonious characterization of ω-categorical theories:
every infinite homogeneous structure has ω-categorical complete theory.

Classifications of some homogeneous structures

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1. homogeneous partial orders (Schmerl 1979).
2. homogeneous (undirected) graphs (Gardiner, Golfand – Klin, Sheehan, Lachlan – Woodrow 1974–1980).
3. homogeneous tournaments (Lachlan 1984).
4. homogeneous directed graphs (Cherlin 1998).
5. homogeneous stable V-structures for any finite relational vocabulary V (Lachlan, Cherlin... 80ies).
6. homogeneous multipartite graphs (Jenkinson, Truss, Seidel 2012).

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Note: The case 4 contains *uncountably* many structures, by a well-known result of Henson (1972).

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- T is **supersimple** $\iff$ the **SU-rank is ordinal valued** for every type of T, and
- T is **1-based** $\iff$ the notion of **(in)dependence** behaves **"nicely"** on all models of T.
- T has **trivial dependence** if whenever $\mathcal{M} \models T$, $A, B, C \subseteq \mathcal{M}^{\mathrm{eq}}$ ($\mathcal{M}$ extended by imaginaries) and A is dependent on B over C, then there is $b \in B$ such that A is dependent on $\{b\}$ over C.

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An **infinite structure** is ω-**categorical, simple, supersimple, 1-based** or **has trivial dependence** if its complete theory has the corresponding property.

Simple theories/structures (continued)

The SU-rank of a structure is the supremum (if it exists) of the SU-ranks of all 1-types of its complete theory.

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Two facts:

1. If T is ω-categorical and supersimple with finite SU-rank, then T is 1-based if and only if every definable (with parameters) $A \subseteq M^{\mathrm{eq}}$ with SU-rank 1 is 1-based for any choice of $\mathcal{M} \models T$.
2. If $\mathcal{M}$ is homogeneous, simple and 1-based, then it is supersimple with finite SU-rank and has trivial dependence.

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In fact we have:

Theorem 1 [2] *Suppose that $\mathcal{M}$ is a structure which is binary, simple and homogeneous. Then $\mathcal{M}$ is* **super***simple with* **finite** *SU-rank (which is bounded by the number of 2-types over $\emptyset$).*

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This implies that knowledge about properties such as 1-basedness and trivial dependence for binary simple homogeneous structures can be derived from the corresponding properties of definable sets of SU-rank 1.

Binary random structures

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$\mathcal{M}$ is a **binary random structure** if it does **not** have a minimal forbidden configuration of cardinality ≥ 3.

Example: random graph.

Roughly speaking: $\mathcal{M}^{\mathrm{eq}}$ is the extension of $\mathcal{M}$ with **imaginary elements**, i.e. elements that correspond to equivalence classes of $\emptyset$-definable equivalence relations on M^n (for $0 < n < \omega$).

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Suppose that $A \subseteq M$ and that $C \subseteq M^{\mathrm{eq}}$ is A-definable (i.e. definable with parameters from A).

The **canonically embedded structure of $\mathcal{M}^{\mathrm{eq}}$ over A with universe** C is the structure $\mathcal{C}$ which for every $0 < n < \omega$ and A-definable relation $R \subseteq C^n$ has a relation symbol which is interpreted as R (and $\mathcal{C}$ has no other symbols).

Note that for all $0 < n < \omega$ and all $R \subseteq C^n$,
R is $\emptyset$-definable in $\mathcal{C} \iff R$ is A-definable in $\mathcal{M}^{\mathrm{eq}}$.

Simple homogeneous structures and canonically embedded structures with rank 1

Let $\mathcal{M}$ and $\mathcal{N}$ be two structures which need **not** necessarily have the same vocabulary.

$\mathcal{N}$ is a **reduct** of $\mathcal{M}$ if $M = N$ and for all $0 < n < \omega$ and all $R \subseteq M^n$: R is $\emptyset$-definable in $\mathcal{N} \implies R$ is $\emptyset$-definable in $\mathcal{M}$.

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Theorem 2 [3] *Suppose that $\mathcal{M}$ is a binary, homogeneous, simple structure with trivial dependence. Let $A \subseteq M$ be finite and suppose that $C \subseteq M^{\mathrm{eq}}$ is A-definable and only finitely many 1-types over $\emptyset$ are realized in C. Assume that $\mathrm{SU}(c/A) = 1$ for every $c \in C$, where $\mathrm{SU}(a/A)$ is the SU-rank of the type tp(c/A). Let $\mathcal{C}$ be the canonically embedded structure of $\mathcal{M}^{\mathrm{eq}}$ over A with universe C. Then $\mathcal{C}$ is a reduct of a binary random structure.*

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Note: If $\mathcal{M}$ is homogeneous, simple and 1-based, then dependence is trivial. All known homogeneous simple structures are 1-based.

Homogeneous, simple and 1-based structures

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$\mathcal{M}$ is **primitive** if there is no nontrivial equivalence relation on M which is $\emptyset$-definable.

Fact, which is straightforward to prove: *Suppose that $\mathcal{M}$ is homogeneous (and simple) and has a nontrivial equivalence relation $E \subseteq M^2$. Let N be one of the E-classes. Then the substructure of $\mathcal{M}$ with universe N is homogeneous (and simple).*

Homogeneous, simple and 1-based structures (cont.)

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Theorem 3. [4] *Suppose that $\mathcal{M}$ is a binary, primitive, homogeneous, simple and 1-based structure. Then $\mathcal{M}$ is* **strongly interpretable in a binary random structure***.*

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Theorem 3. [4] *Suppose that $\mathcal{M}$ is a binary, primitive, homogeneous, simple and 1-based structure. Then $\mathcal{M}$ is* **strongly interpretable in a binary random structure**.

That $\mathcal{M}$ **is strongly interpretable in** $\mathcal{N}$ rougly means that there are integers $k_1, \ldots, k_m$ such that every element $a \in M$ can be identified with a k_i-tuple $\bar{b}_a \in N^i$ for some i in such a way that each $\emptyset$-definable relation in $\mathcal{M}$ can be identified with an $\emptyset$-definable relation on tuples form N corresponding to elements in $\mathcal{M}$.

The well-known notion of **interpretability** is a generalization of strong interpretability.

References

[1] Andrés Aranda López, *Omega-categorical Simple Theories*, Ph.D. thesis, University of Leeds, 2014.

The following (submitted) articles can be found via the link `http://www2.math.uu.se/~vera/research/index.html` and on *arXiv*.

[2] V. Koponen, *Binary simple homogeneous structures are supersimple with finite rank*.

[3] Ove Ahlman, V. Koponen, *On sets with rank one in simple homogeneous structures*.

[4] V. Koponen, *Homogeneous 1-based structures and interpretability in random structures*.

More references are found in the sources above.