A joint logic of problems and propositions, a modified BHK-interpretation, and proof-relevant topological models of intuitionistic logic

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Outline

1. Intuitionistic logic as the logic of problem solving
   - Kolmogorov's Program

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2. QHC: A joint calculus of problems and propositions
   - Basic properties
   - Syntactic interpretations

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   - Basic properties
   - Syntactic interpretations
3. Proof-relevant topological models
   - Medvedev–Skvortsov models recovered
4. A modified BHK interpretation of intuitionistic logic

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- But, as a topologist interested in things like higher-dimensional group theory, I can no longer ignore Martin-Löf type theory (and hence also intuitionistic logic) in my non-foundational research.
- However, few people seem to have ever cared to understand intuitionistic logic from a classical perspective that makes sense to a non-foundationally inclined mathematician.

Intuitionistic logic as the logic of problem solving

A. Kolmogoroff, *Zur Deutung der intuitionistischen Logik* (1932):

“On a par with theoretical logic, which systematizes schemes of proofs of theoretical truths, one can systematize schemes of solutions of problems — for example, of geometric construction problems. [...] Thus, in addition to theoretical logic, a certain new *calculus of problems* arises. [...]

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Surprisingly, the calculus of problems coincides in form with Brouwer's intuitionistic logic, as recently formalized by Heyting. [In fact, we shall argue] that [intuitionistic logic] should be replaced with the calculus of problems, since its objects are in reality not theoretical propositions but rather problems."

Kolmogorov's argument for changing the terminology was philosophical; here is a psychological one.

- When including intuitionistic logic in a broader context that also includes classical logic, the words "proposition" and "proof" are already reserved for the classical notions, so one needs new words for the intuitionisic notions.
- The words "problem" and "solution" serve this purpose ideally, that is, in full agreement with conventional mathematical practice.

Example

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- the Law of Excluded Middle would not help a student to solve this problem on an exam (in Galois theory).

Intended meaning of the word “Problem”

“Problem” not as in *open problem*, but as in *chess problem, initial value problem, geometric construction problem*.

Kolmogorov: *Aufgabe* (not *Problem*); *задача* (not *проблема*)

English: *task, assignment, exercise, challenge, aim, mission*.

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Martin-Löf (1984): "The [interpretation of Kolmogorov] is very close to programming. 'a is a method [of solving the problem (doing the task) A]' can be read as 'a is a program [...] which meets the specification A'. In Kolmogorov's interpretation, the word problem refers to something to be done and the word [solution] to how to do it."

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• *Prove that G, H are isomorphic* • *Find an isomorphism $G \to H$*	• *G is isomorphic to H*

(depending on formalization, one proof might correspond to several isomorphisms or to no explicit isomorphism)

Kolmogorov's letter to Heyting

Kolmogorov's 1931 letter to Heyting (published in 1988):

"Each 'proposition' in your framework belongs, in my view, to one of two sorts:

(α) p expresses hope that in prescribed circumstances, a certain experiment will always produce a specified result. [...]

(β) p expresses the intention to find a construction. [...]

I prefer to keep the name *proposition* (Aussage) only for propositions of type (α) and to call "propositions" of type (β) simply *problems* (Aufgaben). Associated to a proposition p are the problems $\sim p$ (to derive contradiction from p) and $+p$ (to prove p)."

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Related work: Linear logic, Artëmov (1994–), Japaridze (2002–), Liang–Miller (2012–)

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- Atomic formulas: problem symbols, propositional symbols (possibly depending on variables that all range over the same domain of discourse) and the constant $\bot$

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  Intended reading: $!p$ =*"Prove p"*; $?\alpha$ =*"α has a solution"*
- There are two types of judgements:
  $\vdash \alpha$, with intended meaning *"A solution of α is known"*
  $\vdash p$, with intended meaning *"p is true"*

Problems vs. Propositions (revisited)

Problems		Propositions
• *Prove p*	$\overset{!}{\longleftarrow}$ $\underset{?}{\longrightarrow}$	• *p holds* • *there exists a proof of p*

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QHC calculus: Axioms and inference rules

- All axioms and inference rules of classical predicate calculus applied to propositions (possibly involving ?, !).
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  2. Kreisel's addendum to the BHK; in Kolmogorov's language, "every solution of a problem α should include a proof that it does solve α"
  3. Gödel's axioms of "absolute proofs" — a proof-relevant version of modal axioms of S4 (Lecture at Zilsel's, 1938, published 1995)

The problem solving interpretation (Kolmogorov, 1932)

- a solution of $\alpha \wedge \beta$ consists of a solution of α and a solution of β
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- a solution of $\forall x\, \alpha(x)$ is a general method of solving $\alpha(x_0)$ for all $x_0 \in D$
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As long as intuitionistic logic *per se* is concerned, this is merely a rewording of the so-called "BHK interpretation" (or rather vice versa, historically).

Kreisel's addendum

Schwichtenberg's paradox: The problem

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What is hard is to prove that M actually succeeds on all inputs.

"Kreisel's thesis": every solution of a problem α should include a proof that it does solve α.

(Parallel to Kreisel's addendum to the BHK; arguably implicit in some passages by Kolmogorov.)

Gödel's absolute proofs

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- interpreted, in particular, as the modality $\Box$ of S4
- particular "absolute proofs" interpreted by a proof-relevant version of S4 (also found in the work of Artëmov):
  - $\dfrac{p}{\exists t\ t : p}$
  - $t : (p \to q) \ \to \ (s : p \to t(s) : q)$
  - $t : p \ \to \ p$
  - $t : p \ \to \ t_! : (t : p)$

QHC calculus: New axioms and inference rules

$\neg ?\bot \quad \Leftrightarrow \quad ?\neg\alpha \to \neg ?\alpha \quad \Leftrightarrow \quad \dfrac{\neg\alpha}{\neg ?\alpha}$ (BHK) $\qquad \dfrac{\alpha}{?\alpha}$

$\neg !?\bot \quad \Leftrightarrow \quad !\neg p \to \neg !p \quad \Leftrightarrow \quad \dfrac{\neg p}{\neg !p} \qquad \dfrac{p}{!p}$ (Gödel)

$?(\alpha \to \beta) \;\to\; (?\alpha \to ?\beta)$ (BHK) $\qquad \alpha \to !?\alpha$ (Kreisel)

$!(p \to q) \;\to\; (!p \to !q)$ (Gödel) $\qquad ?!p \to p$ (Gödel)

- $?(\alpha \vee \beta) \;\leftrightarrow\; ?\alpha \vee ?\beta \quad \Leftrightarrow \quad !p \vee !q \;\to\; !(p \vee q)$ (BHK)
- $?\exists x \alpha(x) \;\leftrightarrow\; \exists x ?\alpha(x) \quad \Leftrightarrow \quad \exists x !p(x) \;\to\; !\exists x p(x)$ (BHK)
- $?(\alpha \wedge \beta) \;\leftrightarrow\; ?\alpha \wedge ?\beta \quad \Leftrightarrow \quad !p \wedge !q \;\leftrightarrow\; !(p \wedge q)$ (BHK)
- $?\forall x \alpha(x) \;\to\; \forall x ?\alpha(x) \quad \Leftrightarrow \quad \forall x !p(x) \;\leftrightarrow\; !\forall x p(x)$ (BHK)

(the last two lines and all $\leftarrow$ arrows are redundant)

QHC calculus: Axioms and inference rules (cont'd)

Arguably the most controversial axiom is "soundness": a proof of falsity leads to absurdity.

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- $!?\bot \to \bot$

This can be seen as a strong form of internal provability of consistency ($0 = ?\bot$):

$$?!(?!0 \to 0)$$

which itself does not need the soundness axiom (just like in S4).

Intuitionistic $\neg$ explained via classical $\neg$

The following is proved using (inter alia) the soundness axiom:

$$\neg\alpha \;\leftrightarrow\; !\neg?\alpha$$

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"We note that $\neg a$ should not be understood as the problem 'prove insolubility of a'. In general, if 'insolubility of a' is considered as a fully definite notion, we only obtain that $\neg a$ implies insolubility of a, but not vice versa."

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Heyting (1934):
being aware of the cited passage, refers to a solution of $\neg a$ as a "proof of impossibility to solve a"

Galois connection

An easy consequence of the axioms:

$$\frac{\alpha \to \beta}{?\alpha \to ?\beta} \qquad\qquad \frac{p \to q}{!p \to !q}$$

Thus ? and ! descend to monotone (=order-preserving) maps between the Lindenbaum ~~algebras~~ posets of QC and QH.

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In other words, these two monotone maps constitute a pair of adjoint functors when the two posets are regarded as categories.

Corollary: Up to equivalence, $!p$ is the easiest among all problems α such that $?\alpha \to p$; and $?\alpha$ is the strongest among all propositions p such that $\alpha \to !p$.

?! and !? as modalities

Another corollary: $\Box := ?!$ induces an interior operator on the Lindenbaum poset of QC; and $\nabla := !?$ induces a closure operator on the Lindenbaum poset of QH. That is,

- $\Box p \to p$
- $\Box p \to \Box\Box p$
- $\dfrac{p \to q}{\Box p \to \Box q}$

- $\alpha \to \nabla\alpha$
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The last line is actually a consequence of stronger properties.

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- $\dfrac{p}{\Box p}$
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- $\dfrac{\neg\alpha}{\neg\nabla\alpha}$
- $\nabla(\alpha \to \beta) \to (\nabla\alpha \to \nabla\beta)$
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$+ \text{QH} = \text{QH4}$

QH4\#1: Goldblatt, *Grothendieck topology as geometric modality* (1981)

QH4\#4: Artëmov–Protopopescu, *Intuitionistic epistemic logic* (2014)

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+ QC = QS4

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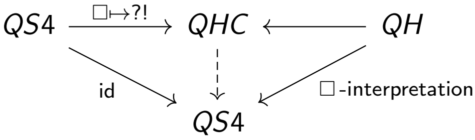
$$QS4 \xrightarrow{\Box\mapsto ?!} QHC \xleftarrow{\nabla\mapsto !?} QH4$$

Interpretations: preserve derivability of formulas and rules.

?!p=*"There exists a proof of p"*; !?α=*"Prove that α has a solution"*

Extension of Gödel's □-interpretation



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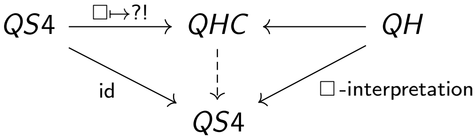

- Atomic problems turn into (new) atomic propositions (including $\bot$, which turns into the classical falsity 0), and get prefixed by $\Box$
- Intuitionistic connectives turn into classical ones and get prefixed by $\Box$ (only $\to$ and $\forall$ really need to be prefixed)
- ? is erased, and ! is replaced by $\Box$

Proposition: This is a (sound) interpretation of QHC in QS4, extending the Gödel □-translation and fixing QC.

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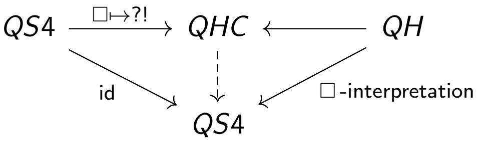

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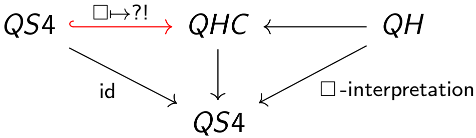
Proposition: This is a (sound) interpretation of QHC in QS4, extending the Gödel $\Box$-translation and fixing QC.

Corollary 1: QHC is sound.

Extension of Gödel's □-interpretation



- Atomic problems turn into (new) atomic propositions (including $\bot$, which turns into the classical falsity 0), and get prefixed by $\Box$
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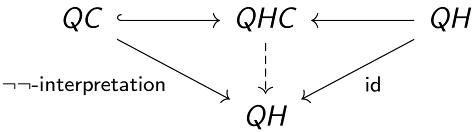
Extension of Kolmogorov's ¬¬-interpretation



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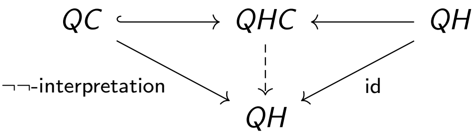

- Atomic propositions turn into (new) atomic problems and get prefixed by ¬¬
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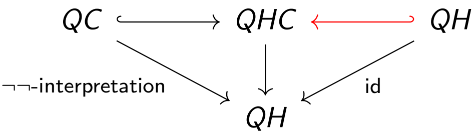

- Atomic propositions turn into (new) atomic problems and get prefixed by $\neg\neg$
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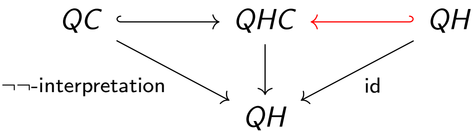

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Question: Is it a conservative extension of QH4?

Degenerate topological models

$$\text{QHC} \xrightarrow{\Box\text{-interpretation}} \text{QS4} \xrightarrow{\text{Topological model}} \text{subsets of } X$$

- propositions $\mapsto$ arbitrary subsets of X
- problems $\mapsto$ open subsets of X
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Proposition: Out of 11 interesting independent principles for QHC, 4 hold in all $\Box$-models, 6 hold in all $\neg\neg$-models, and one (the ?-principle: $\dfrac{?\alpha}{\alpha}$) holds in both $\Box$- and $\neg\neg$-models.

Lafont's Thesis and proof relevance

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So ∇ is similar to the squashing/bracket operator in type theory: Awodey–Bauer, *Propositions as [types]* (2004).

∇-interpretation of QHC in itself

$$
\begin{array}{ccccc}
QS4 & \xhookrightarrow{\Box\mapsto ?!} & QHC & \hookleftarrow & QH \\
\Big\downarrow \text{id} & & \Big\downarrow & & \Big\downarrow \nabla\text{-translation} \\
QS4 & \xhookrightarrow{\Box\mapsto ?!} & QHC & \xleftarrow[\nabla\mapsto !?]{} & QH4
\end{array}
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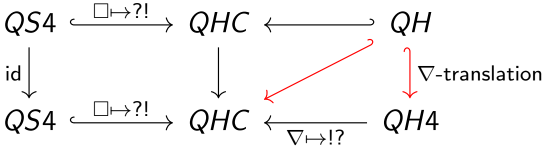

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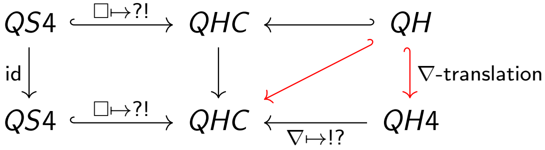

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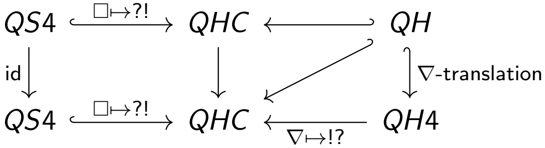

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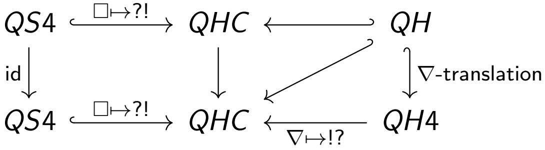

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- LEM for the squashed QH $\Leftrightarrow$ collapse of ∇ (i.e., $\nabla = \neg\neg$).

Proof-relevant topological models

QHC leads to a new paradigm that views intuitionistic logic not as an alternative to classical logic that criminalizes some of its principles, but as an extension package that upgrades classical logic without removing it. The main purpose of the upgrade is proof-relevance, or “categorification”.

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Palmgren (2004): models of QH in *LCCCs with finite sums*
(Suddenly) This includes the topos of sheaves

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- $? \mapsto$ "support", $\operatorname{Supp} \mathcal{F} = \{b \in B \mid \mathcal{F}_b \neq \emptyset\}$
- $! \mapsto$ "characteristic sheaf", $\operatorname{Char} S = (\operatorname{Int} S \hookrightarrow B)$

Example

$f : X \to B$ continuous map

Problem: Find a solution of the equation $f(x) = b$

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With this in mind, our problem is essentially a sheaf!

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- $|?(\alpha \vee \beta)| = \mathsf{Supp}\,|\alpha| \sqcup |\beta| = \mathsf{Supp}\,|\alpha| \cup \mathsf{Supp}\,|\beta| = |?\alpha \vee ?\beta|$
- $|?(\alpha \wedge \beta)| = \mathsf{Supp}\,|\alpha| \times |\beta| = \mathsf{Supp}\,|\alpha| \cap \mathsf{Supp}\,|\beta| = |?\alpha \wedge ?\beta|$

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Remark: Each of the 11 principles fails in some sheaf model.

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$Y = \{0, 1\}$ with the Alexandrov topology of the poset $0 < 1$.

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- If $\mathcal{F} = \mathcal{G} \times \mathcal{H}$, then $(\mathcal{F}_1, \mathcal{F}_0) = (\mathcal{G}_1 \times \mathcal{H}_1, \mathcal{G}_0 \times \mathcal{H}_0)$
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- If $\mathcal{F} = Hom(\mathcal{G}, \mathcal{H})$, then $(\mathcal{F}_1, \mathcal{F}_0) = (\mathcal{H}_1^{\mathcal{G}_1}, \{\phi \colon \mathcal{G}_1 \to \mathcal{H}_1 \mid \phi(\mathcal{G}_0) \subset \mathcal{H}_0\})$
- If $\mathcal{F} = \prod_{d \in D} \mathcal{F}^d$, then $(\mathcal{F}_1, \mathcal{F}_0) = (\prod_{d \in D} \mathcal{F}_1^d, \prod_{d \in D} \mathcal{F}_0^d)$
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Recovering Medvedev–Skvortsov models (cont'd)

Medvedev: "finite problems", 1962; Skvortsov: "transfinite problems", 1979

In a model, each atomic formula α_i is assigned a set S_i.

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Given a formula ϕ of QH and a collection C of subsets $T_i \subset S_i$, we parse the connectives of ϕ as above to get from the pairs (S_i, T_i) a pair (S, T_C).

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Clearly, $\vDash \alpha$ if and only if this sheaf $|\alpha|$ has a global section.

Läuchli's models

A refined (permutation-invariant) version of Medvedev–Skvortsov models is due (independently?) to Läuchli (1970) and is a *complete* model of QH.

In the propositional case, Läuchli's models can easily be interpreted in generalized sheaf models over $\hat{P} \cup_{\hat{1}=pt} S^1$, where generalized means that $\perp$ is represented not by the empty sheaf, but by any sheaf that admits a sheaf morphism into any other sheaf used in the model.

(This corresponds to the view that $\perp$ is not necessarily unsolvable, but is the hardest among all problems.)

Theorem: Generalized sheaf models are complete as models of H.

Recovering Medvedev–Skvortsov models (rephrased)

Remark: A pair (S, T) amounts to a set S and a map $f\colon S \to \{0, 1\}$.

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To *solve* such a parametric problem means to find a common solution of its particular instances corresponding to all possible values of the parameters.

The problem solving interpretation (again)

- a solution of $\alpha \wedge \beta$ consists of a solution of α and a solution of β
- a solution of $\alpha \vee \beta$ consists of an explicit choice between α and β along with a solution of the chosen problem
- a solution of $\alpha \to \beta$ is a *reduction* of β to α; that is, a general method of solving β on the basis of any given solution of α
- $\bot$ has no solutions; $\neg\alpha$ is an abbreviation for $\alpha \to \bot$
- a solution of $\forall x\, \alpha(x)$ is a general method of solving $\alpha(x_0)$ for all $x_0 \in D$
- a solution of $\exists x\, \alpha(x)$ is a solution of $\alpha(x_0)$ for some explicitly chosen $x_0 \in D$

Problem solving interpretation: The collapse

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Then the problem solving interpretation guarantees that:

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- $[\alpha \vee \beta]$ is the disjoint union $[\alpha] \sqcup [\beta]$
- we have $\Phi\colon [\alpha \to \beta] \to [\beta]^{[\alpha]}$ into the set of all maps
- $[\bot] = \emptyset$
- we have $\Psi\colon [\forall x \alpha(x)] \to \prod_{d \in D}[\alpha(d)]$ into the product
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Let us force Φ to be the identity map. (Why not?) Then

$$[\alpha \vee \neg\alpha] = [\alpha] \sqcup \emptyset^{[\alpha]}$$

is never empty; thus $\alpha \vee \neg\alpha$ has a solution for each problem α.

Problem solving interpretation: The collapse

Thus the problem solving interpretation in itself fails to capture the essence of intuitionistic logic!

Troelstra and van Dalen, *Constructivism in Mathematics,* vol. 1 (1988):

"the BHK-interpretation in itself has no 'explanatory power': the possibility of recognizing a classically valid logical schema as being constructively unacceptable depends entirely on our interpretation of 'construction', 'function', 'operation'."

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Our *modified BHK interpretation* can do the recognition without resorting to any special (computational/epistemic) understanding of functions. "mBHK" (m="mathematical").

Thus the mBHK achieves a goal claimed prematurely by Kolmogorov (1932): "In our setup there is no need for any special, e.g. intuitionistic, epistemic presuppositions."

Towards the mBHK

Our point of departure is Kolmogorov's idea of *general method*:

If $\alpha(x)$ is a problem depending on the variable x "of any sort", then "to present a general method of solving $\alpha(x)$ for every particular value of x" means, according to Kolmogorov, "to be able to solve $\alpha(x_0)$ for every given specific value of x_0 of the variable x by a finite sequence of steps, known in advance (i.e. before the choice of x_0)".

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This roughly corresponds to the notion of a "construction" advocated by Brouwer and Heyting, but is perhaps less rhetorical in that it puts emphasis on the fully rigorous matter of the order of quantifiers.

Towards the mBHK (cont'd)

Next, observe that in our sheaf model of QH, the stalks of sheaves over a point *a* behave precisely according to the BHK:

- $|\alpha \wedge \beta|_a$ is the product of sets $|\alpha|_a \times |\beta|_a$
- $|\alpha \vee \beta|_a$ is the disjoint union $|\alpha|_a \sqcup |\beta|_a$
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The sheaves themselves do not really behave in full accordance with the BHK.

Then, one cannot help suspecting that the BHK is only a locally accurate understanding of intuitionistic logic.

Towards the mBHK (cont'd)

Kolmogorov knew this: he defined $\vdash p$, where p is a formula of the calculus of problems (i.e., intuitionistic logic) built out of problem symbols $a, b, c, \dots$, as the following problem:

"Find a general method of solving the problem $p(a, b, c, \dots)$ *for every particular choice of the problems* $a, b, c, \dots$*."*

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"Find a general method of solving the problem $p(a, b, c, \dots)$ for every particular choice of the problems $a, b, c, \dots$."

The mBHK attempts to simplify and clarify this well-forgotten definition so as to avoid the constructive quantification over all particular problems from "all concrete areas of mathematics".

Compare:

- a discussion of Laüchli's models by Lipton and O'Donnell (1995)
- Martin-Löf's formal spaces (1991) and Ranta's possible worlds (1995)

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Thus, $\alpha \vee \neg\alpha$ may well have a special solution for each particular value of the parameter, and at the same time there might not exist any general method of solving all these instances at once.

Examples: Medvedev–Skvortsov problems; results on algorithmic undecidability.