The Expressive Power of k-ary Inclusion and Exclusion Atoms

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I denote Inclusion Logic containing at most k-ary atoms by INC[k]. Similarly I use EXC[k] and INEX[k] for Exclusion Logic and Inclusion-Exclusion Logic.

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I use DEP[k] and NDEP[k] for Dependence and Nondependence Logics containing at most k-ary atoms.

Some known facts about the expressive power of logics

Fact 1 (Galliani) On the level of formulas:

EXC ≡ DEP

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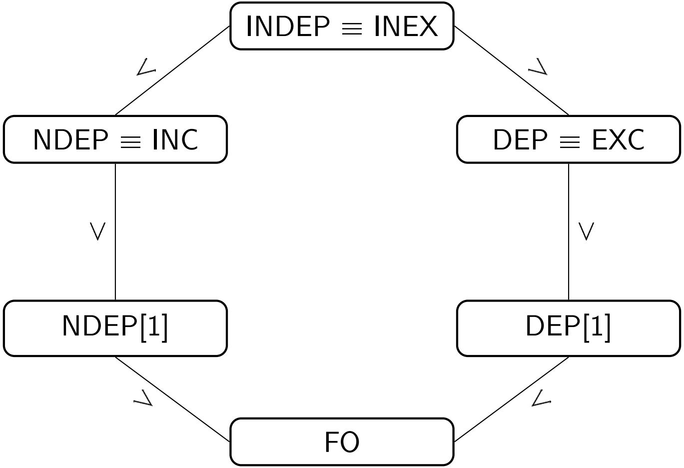
Fact 4 (Galliani and Hella) On the level of sentences:
INC ≡ GFP^+ (Positive Greatest Fixed Point Logic).

The expressive power of logics on the level of formulas



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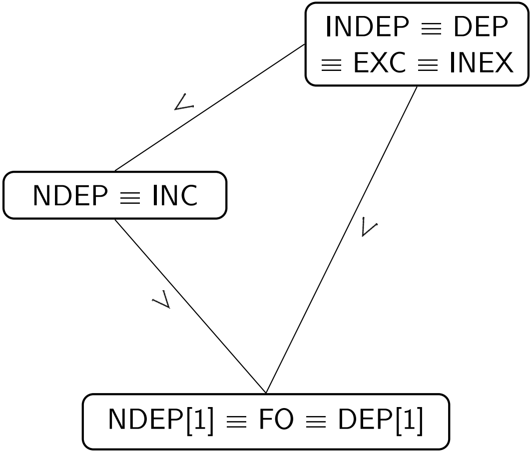

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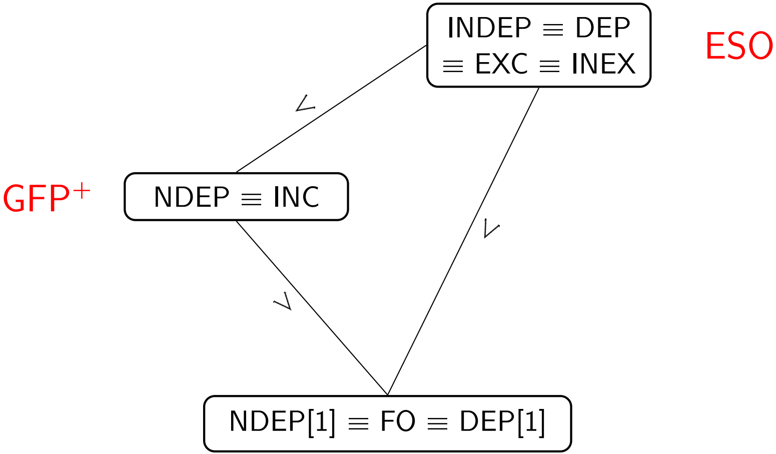

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Corollary DEP[k] $\leqslant$ EXC[k] $\leqslant$ DEP[$k+1$].

Expressing EXC[k] with ESO[k] (k-ary Existential Second Order Logic)

Let φ be EXC[k]-sentence. We label all the instances of exclusion atoms $(\overrightarrow{t_1} \mid \overrightarrow{t_2})_1, \ldots, (\overrightarrow{t_1} \mid \overrightarrow{t_2})_n$ occuring in φ. Now it holds:

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Corollary EXC[k] $\leqslant$ ESO[k].

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The following properties of graphs can be defined in $\mathrm{EXC}[1]$:

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Translation between NDEP and INC

k-ary nondependence atoms can be expressed in INC$[k]$ (Galliani's translation):

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k-ary inclusion atoms can be expressed in NDEP[$k+1$]:

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Corollary NDEP[k] $\leqslant$ INC[k] $\leqslant$ NDEP[$k+1$].

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(\psi \vee \theta)'_i &= \begin{cases} \psi'_i, & \text{if } (t_1 \subseteq t_2)_i \text{ is a subformula of } \psi\\ \theta'_i, & \text{if } (t_1 \subseteq t_2)_i \text{ is a subformula of } \theta\\ \psi'_i \vee \theta'_i, & \text{if } (t_1 \subseteq t_2)_i \text{ is not a subformula of } \psi \vee \theta \end{cases}\\
(\exists x\, \psi)'_i &= \exists x\, \psi'_i\\
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Corollary INC[k] $\leqslant$ ESO[k].

The expressive power of unary inclusion logic

We have shown before that $\mathrm{NDEP}[1] \leqslant \mathrm{INC}[1] \leqslant \mathrm{NDEP}[2]$.

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Expressing INEX[k] with ESO[k]

Let φ be INEX[k]-sentence. We label all the instances of exclusion atoms $(\vec{t_1} \mid \vec{t_2})_1, \ldots, (\vec{t_1} \mid \vec{t_2})_n$ occuring in φ, and define φ':

$$
\begin{aligned}
(\psi)' &= \psi, \text{ if } \psi \text{ is a literal} \\
((\vec{t_1} \mid \vec{t_2})_i)' &= P_i \vec{t_1} \wedge \neg P_i \vec{t_2} \text{ for all } i \in \{1, \ldots, n\} \\
(\vec{t_1} \subseteq \vec{t_2})' &= \vec{t_1} \subseteq \vec{t_2} \\
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Because φ' is an INC[k]-sentence, it is equivalent with some ESO[k]-sentence μ. Now it holds: $\mathcal{M} \models \varphi$, iff $\mathcal{M} \models \exists P_1 \ldots \exists P_n\, \mu$.

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Corollary INEX[k] $\leqslant$ ESO[k].

Term value preserving disjunction

Let $\overrightarrow{t_1}, \ldots, \overrightarrow{t_n}$ be k-tuples of terms.

The following connective can be expressed in INEX[k]:

$\mathcal{M} \models_X \varphi \vee_{\overrightarrow{t_1} \ldots \overrightarrow{t_n}} \psi$, iff there exists $Y, Y' \subseteq X$ s.t.

$$Y \cup Y' = X,\ \mathcal{M} \models_Y \varphi \text{ and } \mathcal{M} \models_{Y'} \psi,$$
$$\text{and if } Y, Y' \neq \emptyset,\ \text{then } Y(\overrightarrow{t_i}) = Y'(\overrightarrow{t_i}) = X(\overrightarrow{t_i}) \quad \text{for all } i \in \{1, \ldots, n\}.$$

Term value preserving disjunction

Expressing term value preserving disjunction in INEX[k]:

$$\varphi \vee_{\vec{t_1}\ldots\vec{t_n}} \psi := (\varphi \sqcup \psi) \sqcup \exists\, y_1 \exists\, y_2 \Big({=}(y_1) \wedge {=}(y_2) \wedge y_1 \neq y_2$$

$$\wedge \exists\, x \Big(\bigwedge_{i=1}^{n} (\theta_i \sqcup \theta_i') \wedge \big((x = y_1 \wedge \varphi) \vee (x = y_2 \wedge \psi)\big)\Big)\Big),$$

where for each $i \in \{1, \ldots, n\}$:

$$\theta_i := (x = y_1 \wedge \forall\, \vec{z}\, (\vec{z} \subseteq \vec{t_i})) \vee (x = y_2 \wedge \forall\, \vec{z}\, (\vec{z} \subseteq \vec{t_i}))$$

$$\theta_i' := \exists\, \vec{u}\, \Big(\vec{u} \mid \vec{t_i} \wedge \exists\, \vec{w_1} \exists\, \vec{w_2} \Big(\big((x = y_1 \wedge \vec{w_1} = \vec{t_i} \wedge \vec{w_2} = \vec{u})$$

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(Connective $\sqcup$ is the *intuitionistic disjunction* which can be expressed in DEP[1])

Expressing ESO[k] with INEX[k]

Let $\exists\, P_1 \ldots \exists\, P_n\, \varphi$ be ESO[k]-sentence that is satisfied, iff it is satisfied with non-empty interpretations for $P_1, \ldots, P_n$. Now it holds:

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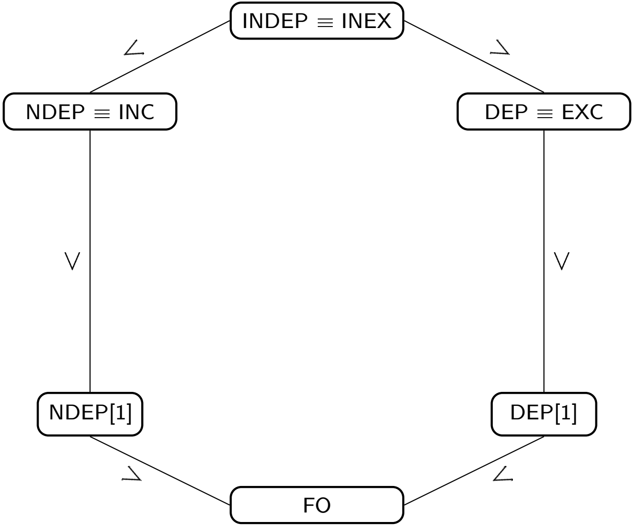
Corollary ESO[k] $\leqslant$ INEX[k], and thus INEX[k] $\equiv$ ESO[k].

The expressive power of logics on the level of formulas



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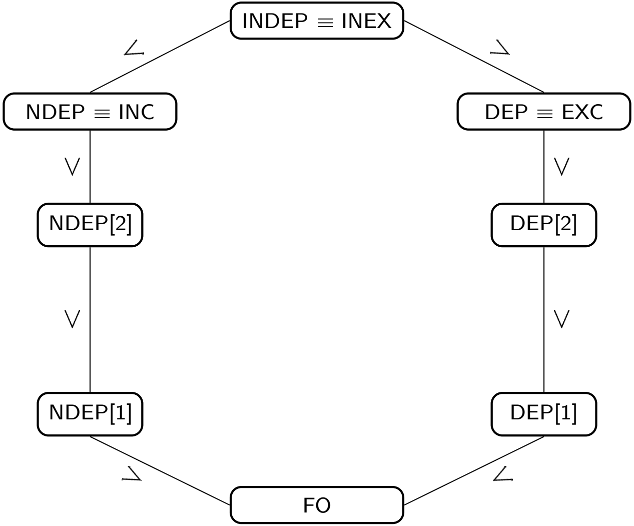

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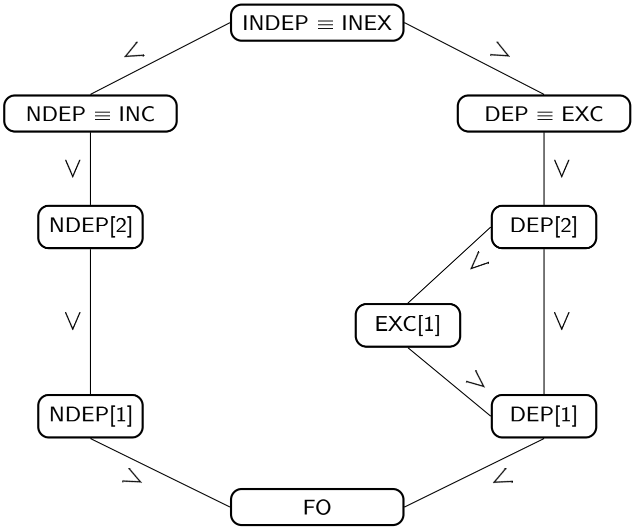

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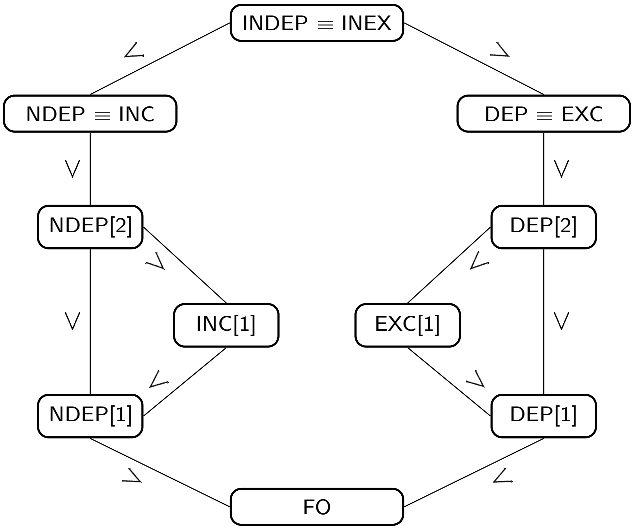

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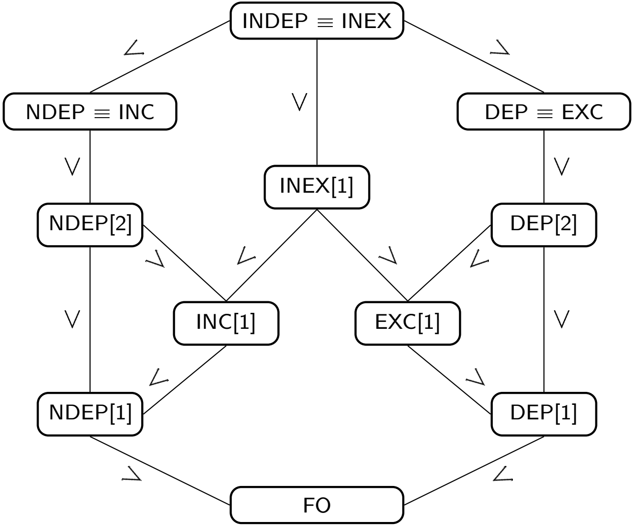

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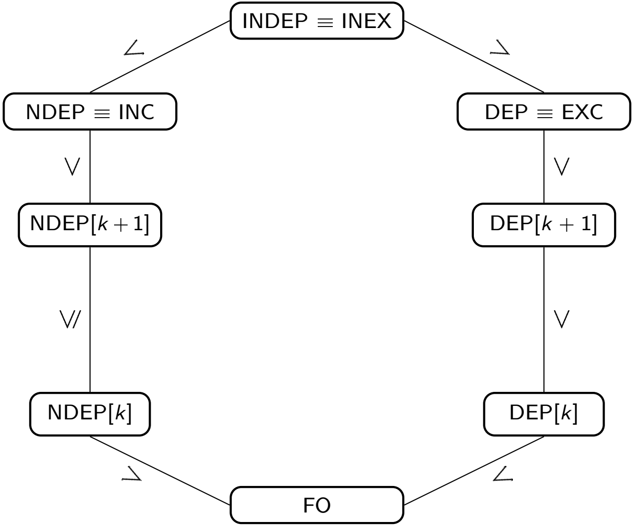

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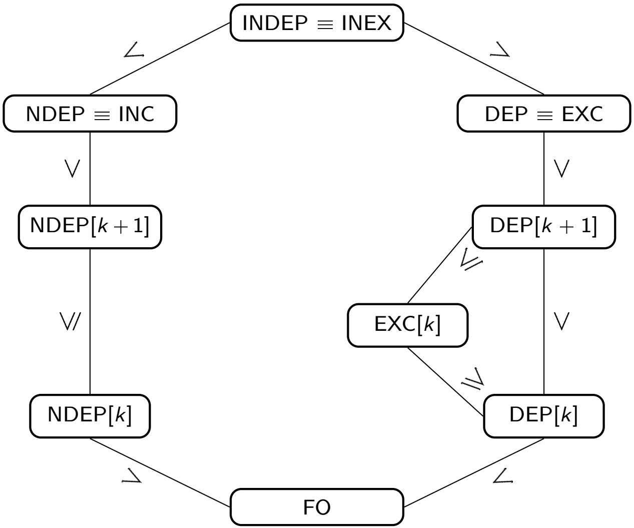

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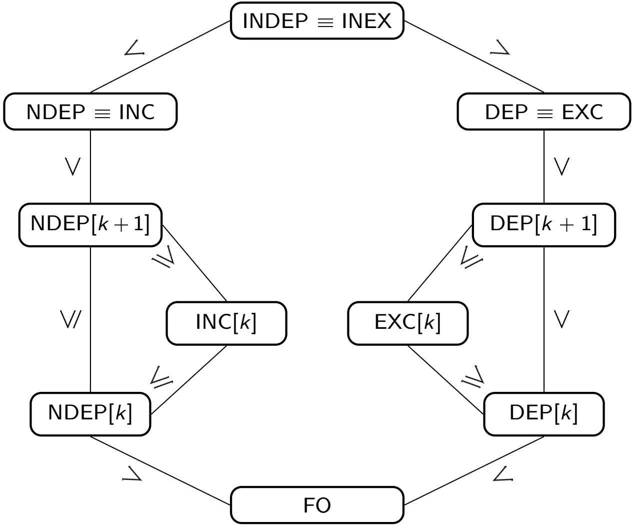

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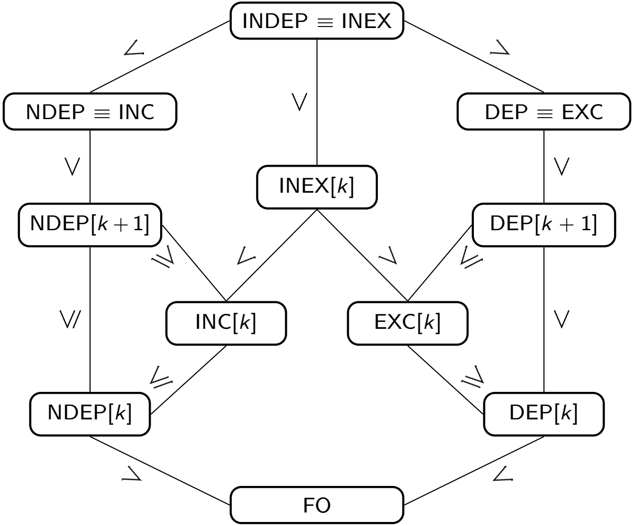

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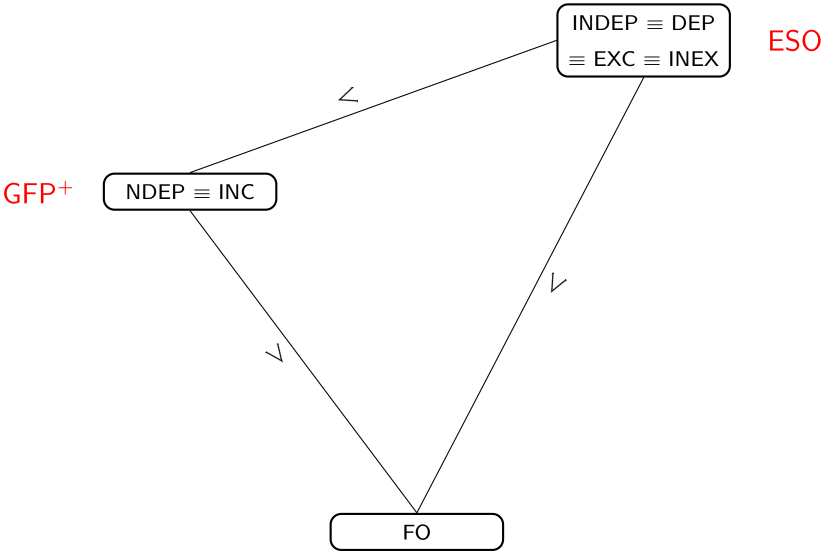

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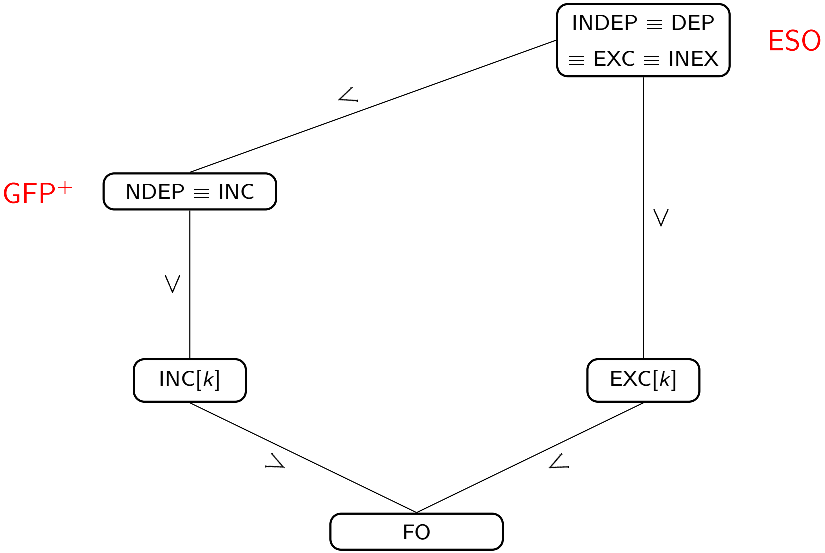

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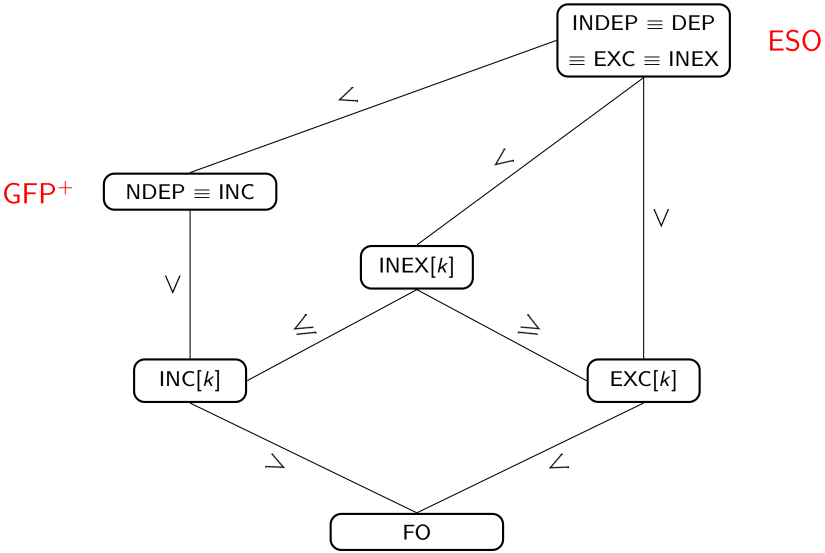

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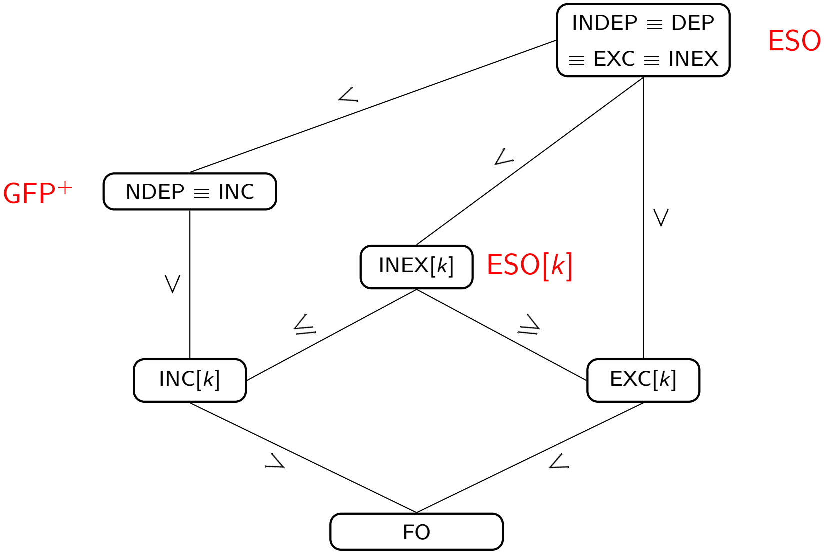

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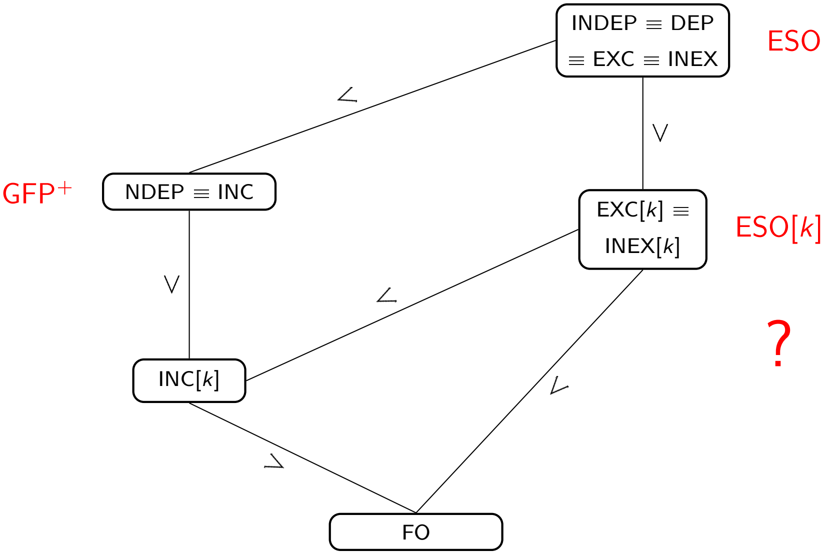

Open questions and further recearch

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- If the latter is true, is it true on the level of sentences?
- What kind of fragments of ESO[k] are INC[k] and EXC[k]?

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Existential quantification over the complement of the values of a given set of terms:

$$\mathcal{M} \models_X (\exists x \mid \bigcup_{i=1}^{n} t_i)\, \varphi,$$

$$\text{iff there exists } f : X \to \overline{\bigcup_{i=1}^{n} X(t_i)}, \text{ s.t. } \mathcal{M} \models_{X[f/x]} \varphi.$$

Appendix: Some properties of graphs expressible in EXC[1]

Undirected graph $\mathcal{G} = (V, E)$ is disconnected, iff

$$\mathcal{G} \models \forall x \, \exists y_1 (\exists y_2 \mid y_1) \big((x = y_1 \lor x = y_2) \\ \land (\forall z_1 \subseteq y_1)(\forall z_2 \subseteq y_2) \neg E z_1 z_2 \big).$$

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Undirected graph $\mathcal{G} = (V, E)$ is k-colorable, iff

$$\mathcal{G} \models \forall x \exists y_1 (\exists y_2 \mid y_1) \dots (\exists y_k \mid y_1 \cup \dots \cup y_{k-1}) \Big(\bigvee_{i=1}^{k} x = y_i \wedge \bigwedge_{i=1}^{k} (\forall z_1 \subseteq y_i)(\forall z_2 \subseteq y_i) \neg E z_1 z_2 \Big).$$

Appendix: A NDEP[2]-formula which is not expressible in INC[1]

Let $L = \emptyset$ and $\mathcal{M}$ be an L-model, s.t. $\text{dom}(\mathcal{M}) = \{0, 1\}$.
Let $X = \{s_1, s_2\}$ and $Y = \{s_1, s_2, s_3\}$, where

$$\begin{aligned} s_1 &= \{(x, 0), (y, 0)\} \\ s_2 &= \{(x, 0), (y, 1)\} \\ s_3 &= \{(x, 1), (y, 1)\} \end{aligned}$$

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Clearly now $\mathcal{M} \models_X \neq\!(x, y)$, but $\mathcal{M} \not\models_Y \neq\!(x, y)$.

Appendix: A NDEP[2]-formula which is not expressible in INC[1]

Let $L = \emptyset$ and $\mathcal{M}$ be an L-model, s.t. $\mathrm{dom}(\mathcal{M}) = \{0, 1\}$.
Let $X = \{s_1, s_2\}$ and $Y = \{s_1, s_2, s_3\}$, where

$$s_1 = \{(x, 0), (y, 0)\}$$
$$s_2 = \{(x, 0), (y, 1)\}$$
$$s_3 = \{(x, 1), (y, 1)\}$$

Clearly now $\mathcal{M} \models_X \neq(x, y)$, but $\mathcal{M} \not\models_Y \neq(x, y)$.

By using the Ehrenfeucht-Fraïssé game for inclusion logic we can show that there exists no INC[1]-formula φ so that

$$\mathcal{M} \models_X \varphi, \text{ but } \mathcal{M} \not\models_Y \varphi.$$