A descriptive set-theoretic view of classification problems in operator algebras
- an overview of recent developments

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I. Prologue
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Recall that a linear operator $T : H \to H$ is **bounded** if

$$\| T \| = \sup \{ \|Tv\| : \|v\| \leq 1 \} < \infty,$$

and that $T$ is bounded precisely when it is continuous with respect to the norm on $H$. The quantity $\| T \|$ is called the **operator norm** of $T$. 
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The set of bounded (linear) operators on $H$ is denoted $\mathcal{B}(H)$. 
$\mathcal{B}(H)$ is a Banach space (complete normed vector space over $\mathbb{C}$) with the operator norm. Moreover, composition of operators make $\mathcal{B}(H)$ into a Banach algebra.
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But **B(H)** is a special kind of Banach algebra. The “adjoint” of an operator **T**, which is the unique operator **T**\(^*\) satisfying

\[ \langle Tv, w \rangle = \langle v, T^* w \rangle, \]

is an **involution** on **B(H)**, which is linear, but anti-multiplicative: 

\( (TS)^* = S^* T^* \).
This involution (the adjoint operation) satisfies the norm identity

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- Equivalently, a $C^*$-algebra is a Banach algebra with an involution satisfying the $C^*$-identity.
The study of subalgebras of $\mathcal{B}(H)$, which (broadly speaking) is what the field operator algebras is all about, grew out of the study of individual operators $T \in \mathcal{B}(H)$. It turns out that it is often fruitful to look not just at $T \in \mathcal{B}(H)$, but at $C^*(T)$, the $C^*$-algebra generated by $T$. In fact, $C^*(T)$ is often "too small", containing too few of the operators needed for understanding the structure of $T$. What we need is a weaker topology than the norm topology.
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In fact, $C^*(T)$ is often “too small”, containing too few of the operators needed for understanding the structure of $T$. What we need is a weaker topology than the norm topology.
Examples:

If $X$ is a compact Hausdorff space, then $C(X)$, the complex valued continuous functions on $X$, forms a $C^*$-algebra with the sup-norm and pointwise composition. These are prototypical Abelian $C^*$-algebras.

Matrix algebras, $M_n(C)$. 

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Definition

The weak topology on $B(H)$ is the weakest topology making the maps $T \mapsto \langle Tv, w \rangle$ continuous for all $v, w \in H$.

This, it turns out, is just one of many useful topologies that are weaker than the norm topology. But for this talk, it is all we need.

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A **von Neumann algebra** is a weakly closed $\ast$-subalgebra of $\mathcal{B}(H)$, which includes the identity operator $I$. 
For an operator $T \in B(H)$, we let $W^*(T) \subseteq B(H)$ denote the von Neumann algebra generated by $T$. Clearly $C^*(T) \subseteq W^*(T)$; in most interesting cases, the inclusion is strict.
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Shortly after $C^*$- and von Neumann algebras were introduced (by Gelfand, Murray and von Neumann), interest arose in creating a structure theory for these algebras. An important definition is the following:

Definition: A von Neumann algebra $A \subseteq B(H)$ is a factor if the centre of $A$, i.e., $Z(A) = \{ T \in A : (\forall S \in A) ST = TS \}$, consists of scalar multiples of the identity of operator, i.e., $Z(A) = C I$.
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Theorem (Murray-von Neumann, late 1930s)

Every von Neumann algebra can be written uniquely as a direct sum or "direct integral" of factors.

The moral of this seems to be that:

▶ Factors are the building blocks of von Neumann algebras.
▶ Whence our focus should be on understanding factors.
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For better or for worse, this lead to people to try to **classify** (up to $^\ast$-isomorphism) von Neumann factors.
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In the beginning, this might have looked easy because there were only a handful of examples of non-isomorphic factors known. All the same, Murray and von Neumann made rough classification of factors into what they called types.

Initially, there were type I, II and III, but then over time people refined this to have type $\text{I}_n$, $n \in \{1, 2, 3, \ldots, \infty\}$, type $\text{II}_1$ and type $\text{II}_\infty$, and finally, type $\text{III}_\lambda$, $\lambda \in [0, 1]$. But then, over time, more and more infinite families of strange and wonderful factors were found leaving one to wonder: Is it at all possible to classify factors up to isomorphism?
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II. Classification problems from the point of view of Descriptive set theory
Recall that descriptive set theory is (roughly speaking) the study of definable sets and functions in and on Polish spaces.

Recall:

- A **Polish space** is a completely metrizable separable topological space.
- A **standard Borel space** is a Borel space where the $\sigma$-algebra is generated by the open sets of some Polish topology on the space.
- A function $f : X \to Y$ between Polish (or standard Borel) spaces $X$ and $Y$ is Borel if $f^{-1}(A)$ is Borel for all Borel $A \subseteq Y$.
  
  Equivalently: The graph of $f$ is Borel in $X \times Y$.
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- etc., etc., etc.
The set $\mathcal{B}(H)$ of bounded linear operators on a separable, complex infinite-dimensional Hilbert space $H$ is not a Polish space in the norm topology (it is not separable), but:

- Recall that the weak topology is the weakest topology making the maps $T \mapsto \langle Tv, w \rangle$ continuous for all $v, w \in H$.
- The weak operator topology is not Polish, but the Borel structure generated by this topology is standard Borel after all.
- We give $\mathcal{B}(H)$ the Borel structure generated by the weakly open sets.
Bounded operators as a standard Borel space

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Borel reducibility

Descriptive set theory can provide a framework for studying classification problems from a **global** point of view.

The central concept is called *Borel reducibility*:

Definition

Let \( X \) and \( Y \) be standard Borel spaces; \( E \) an equivalence relation on \( X \); \( F \) an equivalence relation on \( Y \). A **Borel reduction** of \( E \) to \( F \) is a Borel function \( \theta : X \to Y \) such that \((\forall x, x' \in X)\, x E x' \iff \theta(x) F \theta(x')\). If there is a Borel reduction of \( E \) to \( F \), then we say \( E \) is **Borel reducible** to \( F \), written \( E \leq_B F \).
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We think of the points of $X$ and $Y$ as being interesting objects, or at least "codes" for interesting objects.

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▶ The class of Borel functions plays the role of a suitably (very) general class of “calculable” functions.
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The requirement that $\theta$ be Borel in the definition reflects that to have a “true classification”, the assignment of invariants must be somehow “computable” or “calculable”.

- The class of Borel functions plays the role of a suitably (very) general class of “calculable” functions.
- If we don’t make any assumptions on the definability of the reduction $\theta$, then reducibility would just amount to comparing the cardinality of the quotient spaces $X/E$ and $Y/F$. 
Let again \( H \) be a separable complex Hilbert space.
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- We equip $\Gamma \overset{\text{def}}{=} \mathcal{B}(H)^\mathbb{N}$ with the product Borel structure, which is also standard.

- Given a sequence $\gamma \in \Gamma$, we let:
  - $C^*(\gamma)$ denote the $C^*$-algebra generated by $\gamma$. That is, $C^*(\gamma)$ is the smallest operator norm closed $*$-subalgebra of $\mathcal{B}(H)$ containing $\{\gamma_i : i \in \mathbb{N}\}$.

- We define in $\Gamma$ the equivalence relation $\gamma \equiv C^*(\delta) \iff C^*(\gamma)$ is isomorphic to $C^*(\delta)$.
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In a similar fashion we define, also in $\Gamma = B(H)^\mathbb{N}$:

$\triangleright \ W^*({\gamma})$ denote the von Neumann algebra generated by $\{\gamma_i : i \in \mathbb{N}\}$.

That is, $W^*({\gamma})$ is the smallest weakly closed unital $^*$-subalgebra of $B(H)$ containing $\{\gamma_i : i \in \mathbb{N}\}$.

$\triangleright \ We define in $\Gamma$ the equivalence relation $\gamma \simeq W^*({\delta}) \iff W^*({\gamma})$ is isomorphic to $W^*({\delta})$.

Remark: There is another (equivalent) parametrization as a standard Borel space for the separably acting von Neumann algebras, namely the Effros Borel space. We will return to this if time allows at the end of the talk.
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- $\mathcal{W}^*(\gamma)$ denote the von Neumann algebra generated by $\gamma$. That is, $\mathcal{W}^*(\gamma)$ is the smallest \textbf{weakly} closed \textbf{unital} $*$-subalgebra of $B(H)$ containing $\{\gamma_i : i \in \mathbb{N}\}$. 

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This places the isomorphism relation for separable C*-algebras and separably acting von Neumann algebras within the context of descriptive set theory.
This places the isomorphism relation for separable $C^*$-algebras and separably acting von Neumann algebras within the context of descriptive set theory.

**Basic fact:** The equivalence relations $\sim^{C^*}$ and $\sim^{W^*}$ are analytic as subsets of $\Gamma \times \Gamma$. (i.e., there are Borel functions from $\mathbb{N}^\mathbb{N}$ onto them.)
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**Basic fact:** The equivalence relations $\simeq^{C^*}$ and $\simeq^{W^*}$ are analytic as subsets of $\Gamma \times \Gamma$. (i.e., there are Borel functions from $\mathbb{N}^\mathbb{N}$ onto them.)

**N.b.!** This fact doesn’t rule out that $\simeq^{C^*}$ and $\simeq^{W^*}$ could be Borel. It will follow from later results in this talk that they are in fact not Borel, but are complete analytic.
Other example: The space of countable groups

A different breed of examples comes from *countable structures*. 

A countable group can be thought of as a triple \((f, g, e) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\) such that:

- \(n \cdot f(m) = f(n, m)\) defines a group operation on \(\mathbb{N}\);
- The inverse of \(n\) in this group is given by \(g(n)\);
- \(e\) is the identity element.

Then the set \(\text{P}\) \(= \{ (f, g, e) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : (f, g, e) \text{ defines a group as above} \}\)

is easily seen to be closed in the product topology (taking \(\mathbb{N}\) discrete.)

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  \begin{itemize}
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  \end{itemize}

Then the set $GP = \{ (f, g, e) \in \mathbb{N}^{\mathbb{N} \times \mathbb{N}} \times \mathbb{N}^\mathbb{N} \times \mathbb{N} : (f, g, e) \text{ defines a group as above} \}$ is easily seen to be closed in the product topology (taking $\mathbb{N}$ discrete.).

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Asger Törnquist
A descriptive set-theoretic view of classification problems in operator algebras
An action of the infinite symmetric group

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For \( \delta \in S_\infty \), and \((f, g, e)\) we define

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\delta \cdot f(n, m) = f(\delta^{-1}(n), \delta^{-1}(m)),
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\delta \cdot (f, g, e) = (\delta \cdot f, \delta \cdot g, \delta^{-1}(e))
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is easily seen to induce the isomorphism relation in \( \text{GP} \).
The action of $S_\infty$ on $GP$ is, of course, just a special case of the so-called logic action in model theory.
The Logic Action

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However, the important thing for us is that isomorphism of countable models of a countable language is induced by a natural (and continuous) action of $S_\infty$. 

Note: The “logic actions” are continuous actions of $S_\infty$, so they are Borel.
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Definition
We will say that an action of a Polish group $G$ on a standard Borel space $Y$ is Borel if the map $G \times Y \to Y : (\delta, y) = \delta \cdot y$ is Borel. We will call $Y$ a Borel $G$-space.
The action of \( S_\infty \) on \( \text{GP} \) is, of course, just a special case of the so-called \textbf{logic action} in model theory.

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**Note:** The “logic actions” are continuous actions of \( S_\infty \), so they are Borel.
Each Borel action $a : G \times Y \rightarrow Y$ of a Polish group $G$ on a Polish space $Y$ gives rise to an orbit equivalence relation $E^a$, defined by

$$yE^a y' \iff (\exists g \in G) g \cdot y = y'.$$
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**Note:** The logic action of $S_\infty$ above is Borel, and so the isomorphism relation in $\text{GP}$ is an orbit equivalence relations induced by $S_\infty$. 
Classification by countable structures

Definition

Let \( F \) be an equivalence relation on a standard Borel space \( X \). We will say that \( F \) is classifiable by countable structures if there is a Borel \( S^\infty \)-space \( Y \), with a Borel action \( a: S^\infty \times Y \to Y \), such that \( F \leq BE a \).

Remark: This definition is motivated by the fact that all \( S^\infty \) actions can be described in terms of appropriate "logic actions", for an appropriate choice of structures on \( N \).
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An historical remark, I

The study of the global structure of classification problems essentially goes back to Mackey and his work on unitary representations of groups and \( C^* \)-algebras, which was further developed by Glimm and Effros in the 1960's. The key notion in this work is the smooth/non-smooth dichotomy, which in our terminology is the following:

**Definition**

An equivalence relation \( E \) on a standard Borel space is called **smooth** if there is a Borel reduction of \( E \) to \( = \), the equality relation in \( \mathbb{R} \).

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An historical remark, II

The standard example of an equivalence relation which is not smooth is eventual equality on $2^\mathbb{N} = \{0, 1\}^\mathbb{N}$:

$$x E_0 y \iff (\exists N)(\forall n \geq N) x_n = y_n.$$
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Though $E_0$ is not smooth, it is hardly a horrible equivalence relation. In fact, being able to classify something by using $E_0$ classes as invariants would in most fields of mathematics probably be seen as a victory!
Borel reducibility is a theory that allows us to go far beyond the smooth/non-smooth dichotomy, and prove that naturally occurring equivalence relations are far, far worse than $E_0$. In fact, in most interesting cases, classification problems turn out to be far worse than $E_0$. For instance, already isomorphism of countable graphs or groups is far worse than $E_0$. Comparing classification problems to isomorphism relations of countable structures is a step in the direction of proving that certain classification problems are not just bad, they are worse.
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Comparing classification problems to isomorphism relations of countable structures is a step in the direction of proving that certain classification problems are not just bad, they are worse.
A summary of what has happened so far

Standard Borel spaces may be used to parametrize all separable $C^*$ and von Neumann algebras acting on a separable complex Hilbert space $H$.

There are also standard Borel spaces of "countable structures", such as groups, graphs, but also countable linear orders, hypergraphs, fields, etc.

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Borel reducibility gives us a way of comparing equivalence relations on standard Borel spaces, to "measure their relative complexity".

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Asger Törnquist
A descriptive set-theoretic view of classification problems in operator algebras - an overview of recent developments
Questions going forward

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- Can von Neumann algebras be completely classified by assigning countable groups, graphs or other countable structures as invariants?
- What about C*-algebras?
- If the answer is no, can we make further determinations of “how bad” classification problems are?
III. Applications to classification problems in operator algebras
Von Neumann algebras

As we saw, the primary interest in von Neumann algebras is to classify the so-called factors, which are the building blocks of von Neumann algebras. Attempts at classifying factors suffered a stab to the heart a few years ago:

Theorem (Sasyk-T., 2008)
The isomorphism relation for separably acting factors is not classifiable by countable structures. In fact:

- $\text{II}_1$ factors are not classifiable by countable structures.
- $\text{II}_\infty$ factors are not classifiable by countable structures.
- For each $\lambda \in [0,1]$, the factors of type III$\lambda$ are not classifiable by countable structures.

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**Theorem (Sasyk-T., 2009)**

*ITPFI factors cannot be classified by countable structures.*
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*The isomorphism problem for countable graphs (whence any other kind of countable structure) is Borel reducible to the isomorphism relation for separably acting type $\text{II}_1$ and $\text{II}_\infty$ factors.*
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*In fact, this is true already of group von Neumann algebras of countable discrete icc groups.*
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*The isomorphism problem for countable graphs (whence any other kind of countable structure) is Borel reducible to the isomorphism relation for separably acting type II$_1$ and II$_\infty$ factors.*

In fact, *this is true already of group von Neumann algebras of countable discrete icc groups.*

**Note:** Our proof does not seem to give this for type III$_\lambda$, but it can be derived for type III$_0$ by using a recent result of Foreman and Weiss. For type III$_\lambda$, $\lambda > 0$ it seems to be open.
So the classification of factors is indeed worse than bad. How bad could it be?
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The classification of separable Banach spaces up to linear isomorphism is $\leq_B$ maximal among analytic equivalence relations.
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(Ouch!)
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Conjecture (Törnquist):

The isomorphism relation for separably acting type $\text{II}_1$ factors is $\leq_B$ universal among orbit equivalence relation induced by the unitary group.
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In $C^*$-algebra theory, there is a huge classification program underway since the 1970s for the amenable (i.e., nuclear), simple, separable $C^*$-algebras. It has many successes, but over time it has become clear that the invariants needed seem to grow ever more complex.
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A possible reason is that very complicated invariants are necessary!
Theorem (Farah-Toms-T., 2011)

The isomorphism relation for amenable, simple, separable, unital $C^*$-algebras is not classifiable by countable structures.

The isomorphism relation for countable graphs (and all other types of countable structures) is Borel reducible to isomorphism of amenable, simple, separable, unital $C^*$-algebras.

In fact, the homeomorphism relation for compact metric spaces is Borel reducible to it.
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- In fact, the homeomorphism relation for compact metric spaces is Borel reducible to it.
What about an upper bound? For the nuclear simple separable unital algebras, an upper bound was provided by an action of the automorphism group of $O_2$, but the argument was extremely complicated.
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If the answer to this is no, the most interesting way of answering this is to answer the following:

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