# From Fuzzy Sets to Mathematical Logic

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Zadeh's Fuzzy Set Theory is an important method in dealing with vagueness in applied sciences.

Fuzzy logic in broad sense includes phenomena related to fuzziness and is oriented to real-world applications, while mathematical fuzzy logic develops mathematical methods to model vagueness and fuzziness by well-defined logical tools.

These two approaches do not often meet each other; we try to bridge the gap between practical applications of Fuzzy Set Theory and mathematical fuzzy logic.

Our guiding principle is to explain in logic terms the fuzzy logic concepts that are used in many real world applications, thus we stay as close as possible to practical applications of fuzzy sets.

Our approach is different from the mainstream approach, where the idea is to generalize classical first order logic concepts to many valued logics.

We demonstrate how continuous $[0, 1]$–valued fuzzy sets can be naturally interpreted as open formulas in a simple first order fuzzy logic of Pavelka style. Our main idea is to understand truth values as continuous functions;

- for single elements $x_0 \in X$ the truth values are constant functions defined by the membership degree $\mu_\alpha(x_0)$,
- for open formulas $\alpha(x)$ they are the membership functions $\mu_\alpha : X \curvearrowright [0, 1]$, where the base set $X$ is scaled to the unit interval $[0, 1]$,
- for universally closed formulas $\forall x \alpha(x)$ truth values are definite integrals understood as constant functions. We also introduce existential quantifiers $\exists^a$, where $a \in [0, 1]$.

In the usual mathematical fuzzy logic approaches,

- the truth value of universally closed formulas $\forall x \alpha(x)$ is interpreted via infimum: $v(\forall x \alpha(x)) = \bigwedge_a v(\alpha(a))$.

However, if for all $a$ except one $a_0$,
$v(\alpha(a)) = 1$ and $v(\alpha(a_0) = 0$; then $v(\forall x \alpha(x)) = 0$
and if for all $b$, $v(\alpha(b)) = 0$; then again $v(\forall x \alpha(x)) = 0$.

- the truth value of existentially closed formulas $\exists x \alpha(x)$ is interpreted via supremum: $v(\exists x \alpha(x)) = \bigvee_a v(\alpha(a))$.

However, the condition $v(\exists x \alpha(x)) = b \in [0, 1]$ does not imply that there really would exist some $a$ such that $v(\alpha(a)) = b$.

- Mathematical fuzzy logics based on the above definitions are very close to intuitionistic logic, however, intuitionistic logic is commonly not accepted for the logic of fuzzy phenomena.

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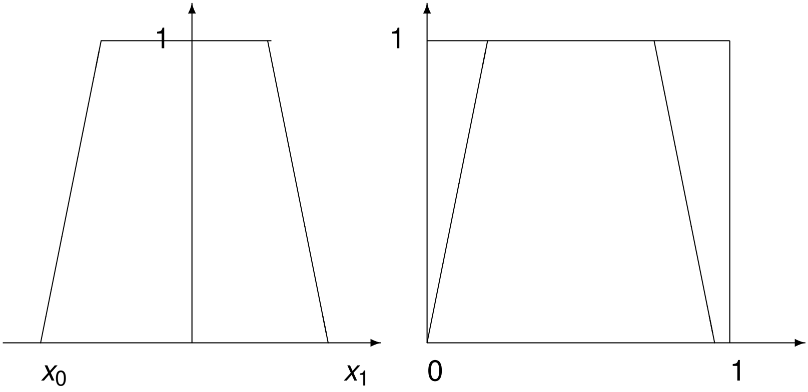
This is our starting point, just a technical detail:

- we will scale $X$ to the interval $[0, 1]$.

A fuzzy set $P(x)$ and its membership function $\overline{P}(x)$ set scaled to $[0, 1]$



In the language under consideration,

- there is a finite number of unary predicates, namely the fuzzy sets $P, R, S, \cdots, T$ and only one free variable $x$; we use notation $P(x), R(x), S(x), \cdots, T(x)$; they are (elementary) open formulas.
- $P(x_0)$, where $x_0 \in [0, 1]$, is a constant formula of the language.
- The logical connectives are $\texttt{or}$, $\texttt{and}$, $\texttt{not}$. For implication connective $\texttt{imp}$ we abbreviate $\alpha \; \texttt{imp} \; \beta := \texttt{not} \; \alpha \; \texttt{or} \; \beta$.
- There is a universal quantifier $\forall$ in the language. If $\alpha(x)$ is an open formula, then $\forall x \alpha(x)$ is a closed formula; read $\forall x \alpha(x)$ 'an average $x$ has a property $\alpha$'. - However, $\texttt{not} \; \forall x \alpha(x)$ is not in the language.

We have the following three principles

- 1. Language and semantics go in hand to hand.
- 2. Truth values are continuous functions
  $v(\alpha) : [0,1] \curvearrowright [0,1]$, denoted by $\overline{\alpha}$ (There is only one valuation!)
- 3. Logical connectives; by the standard MV-operations.

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Thus we define

- for elementary open formulas $A$; $v(A(x)) = \overline{A}(x)$,
- for constant formulas $A(x_0)$, $v(A(x_0)) = \overline{a}(x)$, understood as constant function $\overline{a}(x) \equiv a$ and $\overline{A}(x_0) = a$.
- for formulas closed by the universal quantifier we set

$$v(\forall x \alpha(x)) = \int_0^1 \overline{\alpha}(x) dx = b,$$

where $x$ is free variable in $\alpha$, thus denoted by $\alpha(x)$, and the value $b$ of the definite integral is understood as a constant function $\overline{b} : [0, 1] \curvearrowright [0, 1]$, $\overline{b}(x) \equiv b$.

We define formulas closed by the existential quantifiers $\exists^a$, justified by $x_0 \in [0,1]$. If $v(\alpha(x_0)) = \overline{\alpha}(x_0) = a$, we set

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understood as a constant function $\overline{a}(x) \equiv a$. Thus there are infinitely many existential quantifiers $\exists^a$, one for each $a \in [0,1]$. On the other hand, if there is no such $x_0 \in [0,1]$ that $\overline{\alpha}(x_0) = a$, then $\exists^a x\alpha(x)$ is not defined. - $\mathtt{not}\ \exists^a x\alpha(x)$ is not defined.

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Recall, if $v(\alpha) = \overline{\alpha}$ and $v(\beta) = \overline{\beta}$, then we interpret the logical connectives by point wise defined MV–operations;

$$\begin{aligned} v(\alpha \ \mathtt{and}\ \beta) &= \overline{\alpha} \odot \overline{\beta} &= \max\{\overline{\alpha} + \overline{\beta} - 1, 0\}, \\ v(\alpha \ \mathtt{or}\ \beta) &= \overline{\alpha} \oplus \overline{\beta} &= \min\{\overline{\alpha} + \overline{\beta}, 1\}, \\ v(\mathtt{not}\ \alpha) &= [\overline{\alpha}]^* &= 1 - \overline{\alpha}, \end{aligned}$$

Fundamental: Definite integrals distribute over MV-operations.

Notice that, by using Pavelka style notation, $\models_a \alpha$ has the same meaning than $v(\alpha) = \overline{a}$, where $\overline{a}$ is the membership function – the only truth value – of $\alpha$. Here we list tautologies that are taken schemas for logical axioms. It is a routine task to show that they are 1-tautologies whenever the corresponding formulas are defined

$$
\begin{array}{ll}
(T_1) & \models_1 \alpha \,\mathtt{imp}\, (\mathtt{not}\,\mathtt{not}\,\alpha), \\
(T_2) & \models_1 (\mathtt{not}\,\alpha \,\mathtt{or}\, \mathtt{not}\,\beta) \,\mathtt{imp}\, \mathtt{not}(\alpha \,\mathtt{and}\, \beta), \\
(T_3) & \models_1 (\mathtt{not}\,\alpha \,\mathtt{and}\, \mathtt{not}\,\beta) \,\mathtt{imp}\, \mathtt{not}(\alpha \,\mathtt{or}\, \beta), \\
(T_4) & \models_1 (\mathtt{not}\,\alpha \,\mathtt{or}\, \beta) \,\mathtt{imp}\, (\alpha \,\mathtt{imp}\, \beta), \\
(T_5) & \models_1 (\alpha \,\mathtt{and}\, \mathtt{not}\,\beta) \,\mathtt{imp}\, \mathtt{not}(\alpha \,\mathtt{imp}\, \beta), \\
(T_6) & \models_1 (\mathtt{not}\,\alpha(x_0) \,\mathtt{or}\, \beta) \,\mathtt{imp}\, (\exists^a x \alpha(x) \,\mathtt{imp}\, \beta), \\
 & \quad \text{where } x_0 \text{ justifies } \exists^a \alpha(x), \\
(T_7) & \models_1 (\forall x \,\mathtt{not}\,\alpha(x) \,\mathtt{or}\, \beta) \,\mathtt{imp}\, (\forall x \alpha(x) \,\mathtt{imp}\, \beta).
\end{array}
$$

**Theorem**

*All 1-tautologies of Pavelka's propositional logic are also 1-tautologies in our approach.*

In their seminal book Rasiowa and Sikorski list elementary classical tautologies for quantified formulas, numbered by $(T_{31}) - (T_{61})$. Since $\texttt{not}\,\forall x\alpha(x)$ and $\texttt{not}\,\exists^a x\alpha(x)$ are not formulas in our approach, tautologies $(T_{34}) - (T_{37})$ called *De Morgan laws* are not definable in our language. However,

**Theorem**

*All the classical tautologies that are definable in our approach are 1-tautologies.*

Next we list Pavelka style fuzzy rules of inference to ensure Completeness

Generalized Modus Ponens:

$$\frac{\alpha, \alpha \texttt{ imp } \beta}{\beta} \quad , \quad \frac{\overline{\alpha}, \overline{\gamma}}{\overline{\alpha} \odot \overline{\gamma}}$$

Rule of Bold Conjunction:

$$\frac{\alpha, \beta}{\alpha \texttt{ and } \beta} \quad , \quad \frac{\overline{\alpha}, \overline{\beta}}{\overline{\alpha} \odot \overline{\beta}}$$

Rule of Bold Disjunction:

$$\frac{\alpha, \beta}{\alpha \texttt{ or } \beta} \quad , \quad \frac{\overline{\alpha}, \overline{\beta}}{\overline{\alpha} \oplus \overline{\beta}}$$

Rules for existential quantifiers:

$$\frac{\alpha(x_0)}{\exists^a x \alpha(x)} \quad , \quad \frac{\overline{\alpha}(x_0) = \overline{a} \text{ for some } x_0 \in [0, 1]}{\overline{a}}$$

Rule for universal quantifier:

$$\frac{\alpha(x)}{\forall x \alpha(x)} \quad , \quad \frac{\overline{\alpha}(x)}{\int_0^1 \overline{\alpha}(x) dx}$$

We use Pavelka's definition of graded proof and establish

**Theorem (Soundness and Completeness)**

*If the truth value (i.e. the degree of validity, as there is only one valuation) of a formula $\alpha$ is $\overline{\alpha}$, then there is also an $\mathcal{R}$-proof for $\alpha$ whose value is $\overline{\alpha}$ (by Soundness, this value cannot be greater than $\overline{\alpha}$)*

*Proof.* By induction of the length of formulas.

- We have demonstrated a simple way from fuzzy sets to first order mathematical fuzzy logic.
- The basic idea is to understand the degree of membership as a continuous function.
- Universally closed formulas are then interpreted by definite integrals; this gives the opportunity to define the generalized quantifiers such as almost all, most, many, etc.
- The result is a sound and complete fuzzy logic in Pavelka's sense.
- Our approach is easy to implement e.g. for Matlab or Maple program (work in progress).