The Expressive Power of Modal Dependence Logic

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Motivation and history

Logical modelling of uncertainty, imperfect information and functional dependence in the framework of modal logic.

The ideas are transfered from first-order dependence logic (and independence-friendly logic) to modal logic.

Historical development:

- Branching quantifiers by Henkin 1959.
- Independence-friendly logic by Hintikka and Sandu 1989.
- Compositional semantics for independence-friendly logic by Hodges 1997. (Origin of team semantics.)
- IF modal logic by Tulenheimo 2003.
- Dependence logic by Väänänen 2007.
- Modal dependence logic by Väänänen 2008.

Motivation and history

In IF modal logic, diamonds can be slashed by boxes that precede them: $\Box_1(\Diamond_2/\Box_1)\varphi$.

The idea in modal dependence logic ($\mathcal{MDL}$) is quite different than in IF modal logic: dependences are not between states, but truth values of propositions.

$\mathcal{MDL}$ is not able to express temporal dependencies; to remedy this, Ebbing et al. 2013 introduced extended modal dependence logic ($\mathcal{EMDL}$).

Propositional dependence logic is closely related to the *Inquisitive logic* of Groenendijk 2007.

Syntax for modal logic

Definition

Let Φ be a set of atomic propositions. The set of formulae for standard modal logic $\mathcal{ML}(\Phi)$ is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \Box\varphi,$$

where $p \in \Phi$.

Note that formulas are assumed to be in negation normal form: negations may occur only in front of atomic formulas.

Kripke structures

Definition

Let Φ be a set of atomic propositions. A Kripke model K over Φ is a tuple

$$K = (W, R, V),$$

where W is a nonempty set of *worlds*, $R \subseteq W \times W$ is a binary relation, and V is a *valuation* $V : \Phi \to \mathcal{P}(W)$.

Semantics for modal logic

Definition

Kripke semantics for $\mathcal{ML}$ is defined as follows.

$$
\begin{aligned}
K, w \models p &\quad\Leftrightarrow\quad w \in V(p).\\
K, w \models \neg p &\quad\Leftrightarrow\quad w \notin V(p).\\
K, w \models \varphi \vee \psi &\quad\Leftrightarrow\quad K, w \models \varphi \text{ or } K, w \models \psi.\\
K, w \models \varphi \wedge \psi &\quad\Leftrightarrow\quad K, w \models \varphi \text{ and } K, w \models \psi.\\
K, w \models \Diamond\varphi &\quad\Leftrightarrow\quad K, w' \models \varphi, \text{ for some } w' \text{ s.t. } xRw'.\\
K, w \models \Box\varphi &\quad\Leftrightarrow\quad K, w \models \varphi, \text{ for all } w' \text{ s.t. } xRw'.
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2. The standard semantics for modal logic is given with respect to pointed models K, w. In team semantics the semantics is given for models and teams, i.e., with respect to pairs K, T, where T is a team of K.

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   (a) $K, w \models \varphi$: The actual world is w and φ is true in w.

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   (a) $K, w \models \varphi$: The actual world is w and φ is true in w.
   (b) $K, T \models \varphi$: The actual world is in T, but we do not know which one it is. The formula φ is true in the actual world.
   (c) $K, T \models \varphi$: We consider sets of points as primitive. The formula φ describes properties of collections of points.

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Kripke/Team semantics for $\mathcal{ML}$ is defined as follows. Remember that $K = (W, R, V)$ is a normal Kripke model and $T \subseteq W$.

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K, T \models \Diamond\varphi & \Leftrightarrow & K, T' \models \varphi \text{ for some } T' \text{ s.t.} \\
 & & \forall w \in T\, \exists w' \in T' : wRw' \text{ and } \forall w' \in T'\, \exists w \in T : wRw'.
\end{array}
$$

Note that $K, \emptyset \models \varphi$ for every formula φ.

Team semantics vs. Kripke semantics

Theorem (Flatness property of ML)

Let K be a Kripke model, T a team of K and φ a $\mathcal{ML}$-formula. Then

$$K, T \models \varphi \quad \Leftrightarrow \quad K, w \models \varphi \textit{ for all } w \in T,$$

in particular

$$K, \{w\} \models \varphi \quad \Leftrightarrow \quad K, w \models \varphi.$$

Note that it also follows that every $\mathcal{ML}$-formula is *downwards closed*:

$$\text{If } K, T \models \varphi, \text{ then } K, S \models \varphi \text{ for all } S \subseteq T.$$

Modal dependence logic

Introduced by Väänänen 2008, the syntax modal dependence logic $\mathcal{MDL}$ extends the syntax of modal logic by the clause

$$\mathrm{dep}(p_1, \ldots, p_n, q)\,,$$

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is that the truth value of the propositions $p_1, \dots, p_n$ functionally determines the truth value of the proposition q.

The semantics for $\mathcal{MDL}$ extends the sematics of $\mathcal{ML}$, defined with teams, by the following clause:

$$K, T \models \mathrm{dep}(p_1, \dots, p_n, q)$$

if and only if $\forall w_1, w_2 \in T$:

$$\bigwedge_{i \leq n} \big(w_1 \in V(p_i) \Leftrightarrow w_2 \in V(p_i)\big) \Rightarrow \big(w_1 \in V(q) \Leftrightarrow w_2 \in V(q)\big).$$

Intuitionistic disjunction

$\mathcal{ML}(\ovee)$: add a different version of disjunction $\ovee$ to modal logic with the semantics:

- $K, T \models \varphi \ovee \psi \iff K, T \models \varphi$ or $K, T \models \psi$.

Dependence atoms are definable in $\mathcal{ML}(\ovee)$ (Väänänen 09):

$$K, T \models \mathrm{dep}(p_1, \ldots, p_n, q) \iff K, T \models \bigvee_{s \in F}(\theta_s \wedge (q \ovee \neg q)),$$

where F is the set of all $\{p_1, \ldots, p_n\}$-assignments, and θ_s is the formula $\bigwedge_{i \leq n} p_i^{s(p_i)}$, where $p_i^{\perp} = \neg p_i$ and $p_i^{\top} = p_i$.

Intuitionistic disjunction

It is easy to prove by induction that for every $\mathcal{MDL}$-formula there is an equivalent $\mathcal{ML}(\varovee)$-formula.

Thus, $\mathcal{MDL} \leq \mathcal{ML}(\varovee)$.

However, the converse is not true: There is no formula $\varphi \in \mathcal{MDL}$ that is equivalent with $\Diamond p \varovee \Box\neg p$.

Thus, $\mathcal{MDL} < \mathcal{ML}(\varovee)$.

Extended modal dependence logic $\mathcal{EMDL}$

What is missing from $\mathcal{MDL}$? The counterexample gives a clue: the formula $\Diamond p \otimes \Box\neg p$ is equivalent to $\mathrm{dep}(\Diamond p)$. Thus, we need dependencies between arbitrary modal formulas.

$\mathcal{EMDL}(\Phi)$-formulas are defined by the following grammar:

$$\varphi \quad ::= \quad p \mid \neg\, p \mid \mathrm{dep}(\psi_1, \ldots, \psi_n, \theta) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi,$$

where $p \in \Phi$ and $\psi_1, \ldots, \psi_n, \theta \in \mathcal{ML}$.

The semantics of $\mathrm{dep}(\psi_1, \ldots, \psi_n, \theta)$ is given as for $\mathrm{dep}(p_1, \ldots, p_n, q)$.

With these more general dependence atoms we can express for example temporal dependencies.

Properties of $\mathcal{EMDL}$

Using the idea of Väänänen 09, we can prove that $\mathcal{EMDL}$ is contained in $\mathcal{ML}(\varovee)$:

Theorem (Ebbing, Hella, Meier, Müller, V., Vollmer 13)

$\mathcal{MDL} < \mathcal{EMDL} = \mathcal{ML}(\varovee_{\mathcal{ML}}) \leq \mathcal{ML}(\varovee)$.

($\mathcal{ML}(\varovee_{\mathcal{ML}})$ is the syntactic fragment of $\mathcal{ML}(\varovee)$ in which the clause $\varphi \varovee \varphi$ is applied only to $\mathcal{ML}$-formulae.)

All these logics are downward closed:

Theorem

Let $\varphi \in \mathcal{ML}(\varovee)$. If $K, T \models \varphi$, then $K, S \models \varphi$ for all $S \subseteq T$.

Modal definability and bisimulation

Let $\rightleftarrows_k$ denote the usual k-bisimulation for modal logic.

A class $\mathcal{C}$ of pointed Kripke models (K, w) is *closed under* k*-bisimulation* if it satisfies the condition:

- $(K, w) \in \mathcal{C}$ and $K, w \rightleftarrows_k K', w'$ implies that $(K', w') \in \mathcal{C}$.

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It is well-known that modal definability can be characterized in terms of closure under k-bisimulation:

Theorem (Gabbay, van Benthem)

A class $\mathcal{C}$ of pointed Kripke models is definable in $\mathcal{ML}$ if and only if $\mathcal{C}$ is closed under k-bisimulation for some $k \in \mathbb{N}$.

Team bisimulation

Definition

Let $(K, T), (K', T')$ Kripke models with teams and $k \in \mathbb{N}$. Then K, T and K', T' are *team k-bisimilar*, $K, T \; [\rightleftarrows_k] \; K', T'$, if

1. for every $w \in T$ there is $w' \in T'$ s.t. $K, w \rightleftarrows_k K, w'$, and
2. for every $w' \in T'$ there is $w \in T$ s.t. $K, w \rightleftarrows_k K, w'$.

We say that a class $\mathcal{C}$ of Kripke models with teams is *closed under team k-bisimulation* if it satisfies the condition:

- $(K, T) \in \mathcal{C}$ and $K, T \; [\rightleftarrows_k] \; K', T'$ implies that $(K', T) \in \mathcal{C}$.

The expressive power of $\mathcal{ML}(\varovee)$

Theorem (Hella, Luosto, Sano, V. 14)

A class $\mathcal{C}$ is definable in $\mathcal{ML}(\varovee)$ if and only if $\mathcal{C}$ is downward closed and there exists $k \in \mathbb{N}$ such that $\mathcal{C}$ is closed under team k-bisimulation.

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This result is a natural fusion of the Gabbay – van Benthem characterization for $\mathcal{ML}$, and a corresponding result for the propositional fragment $\mathcal{PL}(\varovee)$ of $\mathcal{ML}(\varovee)$:

Theorem (Ciardelli 09, Yang 14)

All downward closed properties of propositional teams are definable in $\mathcal{PL}(\varovee)$.

The expressive power of $\mathcal{EMDL}$

Remember that $\mathcal{EMDL} \leq \mathcal{ML}(\oslash)$.

Theorem (Hella, Luosto, Sano, V. 14)

$\mathcal{ML}(\oslash) \leq \mathcal{EMDL}$. *Consequently,* $\mathcal{EMDL} \equiv \mathcal{ML}(\oslash)$.

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$\mathcal{ML}(\varobar) \leq \mathcal{EMDL}$. *Consequently,* $\mathcal{EMDL} \equiv \mathcal{ML}(\varobar)$.

Corollary

$\mathcal{ML}(\varobar) \equiv \mathcal{ML}(\varobar_{\mathcal{ML}})$.

Corollary

A class $\mathcal{C}$ *is definable in* $\mathcal{EMDL}$ *iff* $\mathcal{C}$ *is downward closed and there exists* $k \in \mathbb{N}$ *s.t.* $\mathcal{C}$ *is closed under team* k*-bisimulation.*

$\mathcal{EMDL}$ is exponentially more succinct than $\mathcal{ML}(\varovee)$

Theorem (Hella, Luosto, Sano, V. 14)

Let φ be a formula of $\mathcal{ML}(\varovee)$ that is equivalent with $\mathrm{dep}(p_1, \ldots, p_n, q)$. Then $|\varphi| > 2^n$.

Thanks!

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