1.4 Exercises: Some Solutions (November 3, 2011)

1.1. Consider the matrix $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let the elements of $\mathbf{A}$ be independent observations from the random variable x. What is your guess for the rank of $\mathbf{A}$ if x follows

(a) uniform discrete distribution, values being the digits $0, 1, \dots, 9$,
(b) uniform continuous distribution, values being from the interval $[0, 1]$,
(c) Bernoulli distribution: $\mathrm{P}(x = 1) = p$, $\mathrm{P}(x = 0) = 1 - p = q$.

• Solution to Ex. 1.1:

(a) $\mathrm{P}[\mathrm{rk}(\mathbf{A}) = 0] = 1/10^4$,
$\mathrm{P}[\mathrm{rk}(\mathbf{A}) = 1] = \mathrm{P}[\det(\mathbf{A}) = 0] - \mathrm{P}[\mathrm{rk}(\mathbf{A}) = 0] = \text{small}$,
$\mathrm{P}[\mathrm{rk}(\mathbf{A}) = 2] = \text{high}$.

(b) $\mathrm{P}[\mathrm{rk}(\mathbf{A}) = 2] = \text{high}$.

(c) $\mathrm{P}[\mathrm{rk}(\mathbf{A}) = 0] = q^4$,
$\mathrm{P}[\mathrm{rk}(\mathbf{A}) = 1] = p^4 + 4p^2q^2 + 4pq^3$,
$\mathrm{P}[\mathrm{rk}(\mathbf{A}) = 2] = 2p^2q^2 + 4p^3q$. □

1.2. Consider the nonnull vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Prove:

$$\mathbf{P_u P_v} = \mathbf{P_v P_u} \iff \mathbf{u} = \lambda \mathbf{v} \text{ for some } \lambda \in \mathbb{R} \text{ or } \mathbf{u}'\mathbf{v} = 0\,.$$

• Solution to Ex. 1.2:

Denote $\mathbf{u}'\mathbf{u} = a$, $\mathbf{v}'\mathbf{v} = b$ and $\mathbf{u}'\mathbf{v} = c$. Then

$$\begin{aligned} \mathbf{P_u P_v} = \mathbf{P_v P_u} &\iff \frac{\mathbf{uu}'\mathbf{vv}'}{ab} = \frac{\mathbf{vv}'\mathbf{uu}'}{ab} \iff \mathbf{uv}'c = \mathbf{vu}'c \\ &\iff \mathbf{uv}' = \mathbf{vu}' \quad \text{or} \quad c = 0\,. \end{aligned}$$

Postmultiplying $\mathbf{uv}' = \mathbf{vu}'$ by $\mathbf{v}$ yields

$$\mathbf{u} = \frac{\mathbf{vu}'\mathbf{u}}{b} = \frac{a}{b}\mathbf{v} := \alpha \mathbf{v}\,.$$

Trivially $\mathbf{u} = \lambda \mathbf{v}$ for some $\lambda \implies \mathbf{uv}' = \mathbf{vu}'$. □

1.3. Prove Proposition 1.2 (p. 68): $\begin{pmatrix} \mathbf{I}_n \\ -\mathbf{B}' \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{B}_{n \times q} \\ \mathbf{I}_q \end{pmatrix}^{\perp} \right\}$.

• Solution to Ex. 1.3:

Equation

$$\begin{pmatrix} \mathbf{B}_{n\times q} \\ \mathbf{I}_q \end{pmatrix}' \begin{pmatrix} \mathbf{I}_n \\ -\mathbf{B}' \end{pmatrix} = (\mathbf{B}'_{n\times q} : \mathbf{I}_q) \begin{pmatrix} \mathbf{I}_n \\ -\mathbf{B}' \end{pmatrix} = \mathbf{0}$$

implies

$$\mathscr{C}\begin{pmatrix} \mathbf{I}_n \\ -\mathbf{B}' \end{pmatrix} \subset \mathscr{C}\begin{pmatrix} \mathbf{B}_{n\times q} \\ \mathbf{I}_q \end{pmatrix}^{\perp}.$$

It remains to show that the dimensions in the above inclusion are the same. This is true in view of

$$\operatorname{rk}\begin{pmatrix} \mathbf{I}_n \\ -\mathbf{B}' \end{pmatrix} = \operatorname{rk}(\mathbf{I}_n : -\mathbf{B}) = n\,,$$

$$\operatorname{rk}\begin{pmatrix} \mathbf{B}_{n\times q} \\ \mathbf{I}_q \end{pmatrix}^{\perp} = (n+q) - \operatorname{rk}\begin{pmatrix} \mathbf{B}_{n\times q} \\ \mathbf{I}_q \end{pmatrix} = (n+q) - \operatorname{rk}(\mathbf{B}'_{n\times q} : \mathbf{I}_q) = n\,.$$

□

1.4. Show that the matrices $\mathbf{A}$ and $\mathbf{A}'$ have the same maximal number of linearly independent columns.
Hint: Utilize the full rank decomposition of $\mathbf{A}$, Theorem 17 (p. 349).

1.5. Consider an $n \times p$ matrix $\mathbf{A}$ which has full column rank. Confirm that the columns of $\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1/2}$ form an orthonormal basis for $\mathscr{C}(\mathbf{A})$.
Hint: Recall that $(\mathbf{A}'\mathbf{A})^{-1/2} = \mathbf{T}\boldsymbol{\Lambda}^{-1/2}\mathbf{T}'$, where $\mathbf{A}'\mathbf{A} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}'$ is the eigenvalue decomposition of $\mathbf{A}'\mathbf{A}$.

• Solution to Ex. 1.5:

The columns of $\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1/2}$ are orthonormal since

$$\begin{aligned}[\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1/2}]'\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1/2} &= (\mathbf{A}'\mathbf{A})^{-1/2}\mathbf{A}'\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1/2} \\ &= \mathbf{T}\boldsymbol{\Lambda}^{-1/2}\mathbf{T}'\mathbf{T}\boldsymbol{\Lambda}\mathbf{T}'\mathbf{T}\boldsymbol{\Lambda}^{-1/2}\mathbf{T}' = \mathbf{I}_p\,.\end{aligned}$$

Obviously $\mathscr{C}[\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1/2}] = \mathscr{C}(\mathbf{A})$ because for any nonsingular $\mathbf{B}$ we have $\mathscr{C}(\mathbf{A}\mathbf{B}) = \mathscr{C}(\mathbf{A})$. □

1.6. Let $\operatorname{rank}(\mathbf{A}_{n\times p}) = a$ and $\operatorname{rank}(\mathbf{B}_{n\times q}) = b$. Show that the statements

$$\text{(a)}\ \mathscr{C}(\mathbf{A}) \cap \mathscr{C}(\mathbf{B}) = \{\mathbf{0}\} \quad \text{and} \quad \text{(b)}\ \mathscr{C}(\mathbf{A})^{\perp} \cap \mathscr{C}(\mathbf{B})^{\perp} = \{\mathbf{0}\}$$

are equivalent if and only if $\operatorname{rank}(\mathbf{B}) = \operatorname{rank}(\mathbf{A}^{\perp}) = n - \operatorname{rank}(\mathbf{A})$.

1.7. Show that for conformable matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{Z}$ the following holds:

$$\mathscr{C}(\mathbf{A}) \subset \mathscr{C}(\mathbf{B} : \mathbf{Z}) \ \&\ \mathscr{C}(\mathbf{B}) \subset \mathscr{C}(\mathbf{A} : \mathbf{Z}) \iff \mathscr{C}(\mathbf{A} : \mathbf{Z}) = \mathscr{C}(\mathbf{B} : \mathbf{Z}) \,.$$

• SOLUTION TO EX. 1.7:

Suppose the left-hand side is true. Then

$$\begin{aligned}\mathscr{C}(\mathbf{A} : \mathbf{Z}) = \mathscr{C}(\mathbf{A}) + \mathscr{C}(\mathbf{Z}) &\subset [\mathscr{C}(\mathbf{B}) + \mathscr{C}(\mathbf{Z})] + \mathscr{C}(\mathbf{Z}) = \mathscr{C}(\mathbf{B}) + \mathscr{C}(\mathbf{Z}) \\ &\subset [\mathscr{C}(\mathbf{A}) + \mathscr{C}(\mathbf{Z})] + \mathscr{C}(\mathbf{Z}) = \mathscr{C}(\mathbf{A}) + \mathscr{C}(\mathbf{Z}) \,,\end{aligned}$$

and so all inclusions above become equalities; in particular

$$\mathscr{C}(\mathbf{A} : \mathbf{Z}) = \mathscr{C}(\mathbf{B} : \mathbf{Z}) \,.$$

The proof of the reverse relation is obvious. □