2.6 Exercises: Some Solutions (September 13, 2012)

2.1. Prove the equivalence of statements (f), (g) and (h) of Proposition 2.6.

(f) (1) $\mathscr{C}(\mathbf{P}_*) = \mathscr{C}(\mathbf{A})$, (2) $\mathbf{P}_*'\mathbf{V}(\mathbf{I}_n - \mathbf{P}_*) = \mathbf{0}$.

(g) (3) $\mathscr{C}(\mathbf{P}_*) = \mathscr{C}(\mathbf{A})$, (4) $\mathbf{P}_*^2 = \mathbf{P}_*$, (5) $(\mathbf{V}\mathbf{P}_*)' = \mathbf{V}\mathbf{P}_*$.

(h) (6) $\mathscr{C}(\mathbf{P}_*) = \mathscr{C}(\mathbf{A})$, (7) $\mathbb{R}^n = \mathscr{C}(\mathbf{P}_*) \boxplus \mathscr{C}(\mathbf{I}_n - \mathbf{P}_*)$; here $\boxplus$ refers to the orthogonality with respect to the given inner product.

• SOLUTION TO EX. 2.1:

Parts (1) and (2) imply trivially (3) and (5). Writing (2) as

$$\mathbf{P}_*'\mathbf{V} = \mathbf{P}_*\mathbf{V}\mathbf{P}_* = \mathbf{V}\mathbf{P}_* \tag{$*$}$$

and postmultiplying $(*)$ by $\mathbf{P}_*$ yields

$$\mathbf{P}_*'\mathbf{V}\mathbf{P}_* = \mathbf{P}_*\mathbf{V}\mathbf{P}_*^2 = \mathbf{V}\mathbf{P}_*^2$$

which further implies

$$\mathbf{V}\mathbf{P}_* = \mathbf{V}\mathbf{P}_*^2 .$$

Premultiplying the above equation by $\mathbf{V}^{-1}$ gives (4) and so the implication (f) $\implies$ (g) is proved. The reverse relation (g) $\implies$ (f) is easy to confirm.

Part (7) implies that $\mathbf{P}_*'\mathbf{V}(\mathbf{I}_n - \mathbf{P}_*) = \mathbf{0}$ and so (h) $\implies$ (f); thereby also (h) $\implies$ (g). It remains to show that (f) [or equivalently (g)] $\implies$ (h). Now in view of (0.39) (p. 10) the idempotency (4) $\mathbf{P}_*^2 = \mathbf{P}_*$ means that

$$\mathbb{R}^n = \mathscr{C}(\mathbf{P}_*) \oplus \mathscr{C}(\mathbf{I}_n - \mathbf{P}_*) .$$

The condition (2) indicates that above $\oplus$ becomes $\boxplus$. □

2.2. Prove the statements in (2.121) (p. 88).

2.3. [Equality of two projectors under different inner products] Suppose that $\mathbf{V}_1$ and $\mathbf{V}_2$ are positive definite $n\times n$ matrices and $\operatorname{rank}(\mathbf{X}_{n\times p}) = r$. Prove the equivalence of the following statements:

(a) $\mathbf{P}_{\mathbf{X};\mathbf{V}_1^{-1}} = \mathbf{P}_{\mathbf{X};\mathbf{V}_2^{-1}}$, (b) $\mathbf{X}'\mathbf{V}_2^{-1}\mathbf{P}_{\mathbf{X};\mathbf{V}_1^{-1}} = \mathbf{X}'\mathbf{V}_2^{-1}$,

(c) $\mathbf{P}_{\mathbf{X};\mathbf{V}_1^{-1}}'\mathbf{V}_2^{-1}\mathbf{P}_{\mathbf{X};\mathbf{V}_1^{-1}} = \mathbf{V}_2^{-1}\mathbf{P}_{\mathbf{X};\mathbf{V}_1^{-1}}$, (d) $\mathbf{V}_2^{-1}\mathbf{P}_{\mathbf{X};\mathbf{V}_1^{-1}}$ is symmetric,

(e) $\mathscr{C}(\mathbf{V}_1^{-1}\mathbf{X}) = \mathscr{C}(\mathbf{V}_2^{-1}\mathbf{X})$, (f) $\mathscr{C}(\mathbf{V}_1\mathbf{X}^\perp) = \mathscr{C}(\mathbf{V}_2\mathbf{X}^\perp)$,

(g) $\mathscr{C}(\mathbf{V}_2\mathbf{V}_1^{-1}\mathbf{X}) = \mathscr{C}(\mathbf{X})$, (h) $\mathbf{X}'\mathbf{V}_1^{-1}\mathbf{V}_2\mathbf{M} = \mathbf{0}$,

(i) $\mathscr{C}(\mathbf{V}_1^{-1/2}\mathbf{V}_2\mathbf{V}_1^{-1/2} \cdot \mathbf{V}_1^{-1/2}\mathbf{X}) = \mathscr{C}(\mathbf{V}_1^{-1/2}\mathbf{X})$,

(j) $\mathscr{C}(\mathbf{V}_1^{-1/2}\mathbf{X})$ has a basis $\mathbf{U} = (\mathbf{u}_1 : \ldots : \mathbf{u}_r)$ comprising a set of r eigenvectors of $\mathbf{V}_1^{-1/2}\mathbf{V}_2\mathbf{V}_1^{-1/2}$,

(k) $\mathbf{V}_1^{-1/2}\mathbf{X} = \mathbf{U}\mathbf{A}$ for some $\mathbf{A}_{r\times p}$, $\operatorname{rank}(\mathbf{A}) = r$,

(l) $\mathbf{X} = \mathbf{V}_1^{1/2}\mathbf{U}\mathbf{A}$; the columns of $\mathbf{V}_1^{1/2}\mathbf{U}$ are r eigenvectors of $\mathbf{V}_2\mathbf{V}_1^{-1}$,

(m) $\mathscr{C}(\mathbf{X})$ has a basis comprising a set of r eigenvectors of $\mathbf{V}_2\mathbf{V}_1^{-1}$.

Some statements above can be conveniently proved using Proposition 10.1 (p. 222). See also Section 11.1 (p. 275), Exercise 11.9 (p. 288), and (18.81) (p. 388).

Thomas (1968), Harville (1997, p. 265), Tian & Takane (2008b, 2009b), Hauke, Markiewicz & Puntanen (2011).

• Solution to Ex. 2.3:

We have to study the equality

$$\mathbf{X}(\mathbf{X}'\mathbf{V}_1^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}_1^{-1} = \mathbf{X}(\mathbf{X}'\mathbf{V}_2^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}_2^{-1}. \tag{a}$$

Denoting

$$\mathbf{X}_* = \mathbf{V}_1^{-1/2}\mathbf{X}, \quad \mathbf{X} = \mathbf{V}_1^{1/2}\mathbf{X}_*,$$

(a) becomes

$$\mathbf{V}_1^{1/2}\mathbf{X}_*(\mathbf{X}_*'\mathbf{X}_*)^-\mathbf{X}_*'\mathbf{V}_1^{-1/2} = \mathbf{V}_1^{1/2}\mathbf{X}_*(\mathbf{X}_*'\mathbf{V}_1^{1/2}\mathbf{V}_2^{-1}\mathbf{V}_1^{1/2}\mathbf{X}_*)^-\mathbf{X}_*'\mathbf{V}_1^{1/2}\mathbf{V}_2^{-1},$$

which is equivalent to

$$\begin{aligned}\mathbf{X}_*(\mathbf{X}_*'\mathbf{X}_*)^-\mathbf{X}_*' &= \mathbf{X}_*(\mathbf{X}_*'\mathbf{V}_1^{1/2}\mathbf{V}_2^{-1}\mathbf{V}_1^{1/2}\mathbf{X}_*)^-\mathbf{X}_*'\mathbf{V}_1^{1/2}\mathbf{V}_2^{-1}\mathbf{V}_1^{1/2},\\ \mathbf{P}_{\mathbf{X}_*} &= \mathbf{X}_*(\mathbf{X}_*'\mathbf{V}_*^{-1}\mathbf{X}_*)^-\mathbf{X}_*'\mathbf{V}_*^{-1} = \mathbf{P}_{\mathbf{X}_*;\mathbf{V}_*^{-1}},\end{aligned}$$

where

$$\mathbf{V}_* = \mathbf{V}_1^{-1/2}\mathbf{V}_2\mathbf{V}_1^{-1/2}.$$

Now the conditions of Proposition 10.1 (p. 218) for the equality of OLSE and BLUE under the model $\{\mathbf{y}_*, \mathbf{X}_*, \mathbf{V}_*\}$ can be applied; here $\mathbf{y}_* = \mathbf{V}_1^{-1/2}\mathbf{y}$, $\operatorname{cov}(\mathbf{y}) = \mathbf{V}_1$.

We can solve (a) also by considering the linear models $\mathscr{M}_1 = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_1\}$ and $\mathscr{M}_2 = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_2\}$. Now (a) means the equality between the BLUEs under $\mathscr{M}_1$ and $\mathscr{M}_2$. Premultiplying $\mathscr{M}_1$ and $\mathscr{M}_2$ by $\mathbf{V}_1^{-1/2}$ yields the transformed versions of the models $\mathscr{M}_1$ and $\mathscr{M}_2$:

$$\begin{aligned}\mathscr{M}_1^* &= \{\mathbf{V}_1^{-1/2}\mathbf{y}, \mathbf{V}_1^{-1/2}\mathbf{X}\boldsymbol{\beta}, \mathbf{I}_n\} = \{\mathbf{y}_*, \mathbf{X}_*\boldsymbol{\beta}, \mathbf{I}_n\},\\ \mathscr{M}_2^* &= \{\mathbf{V}_1^{-1/2}\mathbf{y}, \mathbf{V}_1^{-1/2}\mathbf{X}\boldsymbol{\beta}, \mathbf{V}_1^{-1/2}\mathbf{V}_2\mathbf{V}_1^{-1/2}\} = \{\mathbf{y}_*, \mathbf{X}_*\boldsymbol{\beta}, \mathbf{V}_*\}.\end{aligned}$$

The equality of the BLUEs is equivalent to the equality of the OLSE and BLUE of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}_2^*$ and hence, e.g., part (v) of Proposition 10.1 gives the condition

$$\mathscr{C}(\mathbf{V}_1^{-1/2}\mathbf{V}_2\mathbf{V}_1^{-1/2} \cdot \mathbf{V}_1^{-1/2}\mathbf{X}) = \mathscr{C}(\mathbf{V}_1^{-1/2}\mathbf{X}), \quad \text{i.e.,} \quad \mathscr{C}(\mathbf{V}_2\mathbf{V}_1^{-1}\mathbf{X}) = \mathscr{C}(\mathbf{X}).$$

i.e., $\mathscr{C}(\mathbf{V}_2\mathbf{V}_1^{-1}\mathbf{X}) = \mathscr{C}(\mathbf{X})$.

One simple way to find conditions for (a) would be to use Theorem 10 (p. 216). Of course, "simplicity" here is based on the use of a powerful result. In any event, suppose that we know that $\mathbf{P}_{\mathbf{X};\mathbf{V}_1^{-1}}\mathbf{y}$ is the BLUE for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}_2$ if and only if

$$\mathbf{P}_{\mathbf{X};\mathbf{V}_1^{-1}}(\mathbf{X} : \mathbf{V}_2\mathbf{M}) = (\mathbf{X} : \mathbf{0}).$$

The $\mathbf{X}$-part above trivially holds and so we are left with

$$\mathbf{X}(\mathbf{X}'\mathbf{V}_1^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}_1^{-1}\mathbf{V}_2\mathbf{M} = \mathbf{0}.$$

Premultiplying the above equation by $\mathbf{X}'\mathbf{V}_1^{-1}$ yields (why?)

$$\mathbf{X}'\mathbf{V}_1^{-1}\mathbf{V}_2\mathbf{M} = \mathbf{0},$$

i.e.,

$$\mathscr{C}(\mathbf{V}_2\mathbf{V}_1^{-1}\mathbf{X}) \subset \mathscr{C}(\mathbf{X}).$$

In view of $\operatorname{rk}(\mathbf{V}_2\mathbf{V}_1^{-1}\mathbf{X}) = \operatorname{rk}(\mathbf{X})$, we have an equality above. □

2.4 (Continued ...). Show that $\mathscr{C}(\mathbf{V}_1\mathbf{X}) = \mathscr{C}(\mathbf{V}_2\mathbf{X}) \nRightarrow \mathscr{C}(\mathbf{V}_1\mathbf{X}^\perp) = \mathscr{C}(\mathbf{V}_2\mathbf{X}^\perp)$, but the following statements are equivalent:

(a) $\mathscr{C}(\mathbf{V}_1\mathbf{X}) = \mathscr{C}(\mathbf{X})$, (b) $\mathscr{C}(\mathbf{V}_1\mathbf{X}^\perp) = \mathscr{C}(\mathbf{X}^\perp)$,
(c) $\mathscr{C}(\mathbf{V}_1^{-1}\mathbf{X}) = \mathscr{C}(\mathbf{X})$, (d) $\mathscr{C}(\mathbf{V}_1^{-1}\mathbf{X}^\perp) = \mathscr{C}(\mathbf{X}^\perp)$.

2.5. Suppose that $\mathbf{V}$ is positive definite $n \times n$ matrix and $\mathbf{X} \in \mathbb{R}^{n \times p}$. Confirm:

$$\begin{aligned}
\mathbf{P}_{\mathbf{X};\mathbf{V}^{-1}} &= \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}^{-1} \\
&= \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1/2}\mathbf{V}^{-1/2}\mathbf{X})^-\mathbf{X}'\mathbf{V}^{-1/2}\mathbf{V}^{-1/2} \\
&= \mathbf{V}^{1/2}\mathbf{P}_{\mathbf{V}^{-1/2}\mathbf{X}}\mathbf{V}^{-1/2} \\
&= \mathbf{V}^{1/2}(\mathbf{I}_n - \mathbf{P}_{(\mathbf{V}^{-1/2}\mathbf{X})^\perp})\mathbf{V}^{-1/2} \\
&= \mathbf{V}^{1/2}(\mathbf{I}_n - \mathbf{P}_{\mathbf{V}^{1/2}\mathbf{M}})\mathbf{V}^{-1/2} \\
&= \mathbf{I}_n - \mathbf{V}\mathbf{M}(\mathbf{M}\mathbf{V}\mathbf{M})^-\mathbf{M} = \mathbf{I}_n - \mathbf{P}'_{\mathbf{M};\mathbf{V}}.
\end{aligned}$$

See also (2.74) (p. 92), Proposition 5.9 (p. 164), and part (i) of Theorem 15 (p. 332).

• SOLUTION TO EX. 2.5:

$$
\begin{aligned}
\mathbf{P}_{\mathbf{X};\mathbf{V}^{-1}} &= \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1} \\
&= \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1/2}\mathbf{V}^{-1/2}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1/2}\mathbf{V}^{-1/2} \\
&= \mathbf{V}^{1/2}\mathbf{V}^{-1/2}\cdot\mathbf{V}^{-1/2}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1/2}\mathbf{V}^{-1/2}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1/2}\cdot\mathbf{V}^{-1/2} \\
&= \mathbf{V}^{1/2}\mathbf{P}_{\mathbf{V}^{-1/2}\mathbf{X}}\mathbf{V}^{-1/2} \\
&= \mathbf{V}^{1/2}(\mathbf{I}_n - \mathbf{P}_{(\mathbf{V}^{-1/2}\mathbf{X})^{\perp}})\mathbf{V}^{-1/2} \\
&= \mathbf{V}^{1/2}(\mathbf{I}_n - \mathbf{P}_{\mathbf{V}^{1/2}\mathbf{M}})\mathbf{V}^{-1/2} \\
&= \mathbf{V}^{1/2}[\mathbf{I}_n - \mathbf{V}^{1/2}\mathbf{M}(\mathbf{M}\mathbf{V}^{1/2}\mathbf{V}^{1/2}\mathbf{M})^{-}\mathbf{M}\mathbf{V}^{1/2}]\mathbf{V}^{-1/2} \\
&= \mathbf{I}_n - \mathbf{V}\mathbf{M}(\mathbf{M}\mathbf{V}\mathbf{M})^{-}\mathbf{M} \\
&= \mathbf{I}_n - \mathbf{P}'_{\mathbf{M};\mathbf{V}}\,.
\end{aligned}
$$

□

2.6. Suppose that $\mathscr{C}(\mathbf{A}) \cap \mathscr{C}(\mathbf{B}) = \{\mathbf{0}_n\} = \mathscr{C}(\mathbf{C}) \cap \mathscr{C}(\mathbf{D})$. Show that then

$$\{\,\mathbf{P}_{\mathbf{C}|\mathbf{D}}\,\} \subset \{\,\mathbf{P}_{\mathbf{A}|\mathbf{B}}\,\} \quad\iff\quad \mathscr{C}(\mathbf{A}) \subset \mathscr{C}(\mathbf{C}) \;\text{ and }\; \mathscr{C}(\mathbf{B}) \subset \mathscr{C}(\mathbf{D})\,.$$

Kala (1981, Lemma 2.5).