3.4 Exercises: Some Solutions (November 26, 2011)

3.1. Let $\mathbf{U} = (\mathbf{1} : \mathbf{x} : \mathbf{y})$ be a 100×3 matrix where $\mathbf{x}$ and $\mathbf{y}$ comprise the observed values of variables x and y, and assume that $s_x^2 > 0$ and $s_y^2 > 0$.

(a) Suppose that the columns of $\mathbf{U}$ are orthogonal. Show that $r_{xy} = 0$.

(b) Confirm that if $\operatorname{rank}(\mathbf{U}) = 2$, then $r_{xy}^2 = 1$.

• Solution to Ex. 3.1:

(a) $\mathbf{U}'\mathbf{U} = \operatorname{diag}(n, \mathbf{x}'\mathbf{x}, \mathbf{y}'\mathbf{y}) \implies \mathbf{1}'\mathbf{x} = \mathbf{1}'\mathbf{y} = 0$ and so $\mathbf{x}$ and $\mathbf{y}$ are centered. Hence $r_{xy} = \mathbf{x}'\mathbf{y}/\sqrt{\mathbf{x}'\mathbf{x} \cdot \mathbf{y}'\mathbf{y}}$, which is 0 because $\mathbf{x}'\mathbf{y} = 0$.

(b) $\operatorname{rank}(\mathbf{U}) < 3 \implies$ there exists $\mathbf{a} = (\alpha, \beta, \gamma)' \neq \mathbf{0}$ such that

$$\mathbf{U}\mathbf{a} = \alpha\mathbf{1} + \beta\mathbf{x} + \gamma\mathbf{y} = 0\,. \tag{$*$}$$

In $(*)$ necessarily $\alpha \neq 0$ and $\beta \neq 0$ because $s_x^2 > 0$ and $s_y^2 > 0$. Hence

$$\mathbf{y} = -\frac{\alpha}{\gamma}\mathbf{1} - \frac{\beta}{\gamma}\mathbf{x} := a\mathbf{1} + b\mathbf{x}\,,$$

i.e., $y_i = a + bx_i, i = 1, \dots, n$, which trivially means that $r_{xy}^2 = 1$. □

3.2. Let $\mathbf{x}$ and $\mathbf{y}$ be n-dimensional variable vectors (comprising the observed values of x an y). Show that

$$\operatorname{cor_d}(a\mathbf{1} + b\mathbf{x}, c\mathbf{1} + d\mathbf{y}) = \begin{cases} \operatorname{cor_d}(\mathbf{x}, \mathbf{y}) & \text{if } bd > 0\,, \\ -\operatorname{cor_d}(\mathbf{x}, \mathbf{y}) & \text{if } bd < 0\,, \end{cases}$$

or in other notation,

$$\operatorname{cor_d}\left[(\mathbf{1} : \mathbf{x})\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), (\mathbf{1} : \mathbf{y})\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)\right] = \begin{cases} r_{xy} & \text{if } bd > 0\,, \\ -r_{xy} & \text{if } bd < 0\,. \end{cases}$$

• Solution to Ex. 3.2:

Denoting $\mathbf{u} = a\mathbf{1} + b\mathbf{x}$, $\mathbf{v} = c\mathbf{1} + d\mathbf{y}$ and $\mathbf{C} =$ the centering matrix, yields

$$\mathbf{u}'\mathbf{C}\mathbf{v} = bd\,\mathbf{x}'\mathbf{C}\mathbf{y}\,, \quad \mathbf{u}'\mathbf{C}\mathbf{u} = b^2\mathbf{x}'\mathbf{C}\mathbf{x}\,, \quad \mathbf{v}'\mathbf{C}\mathbf{v} = d^2\mathbf{y}'\mathbf{C}\mathbf{y}\,,$$

$$\implies r_{uv} = \frac{bd\,\mathbf{x}'\mathbf{C}\mathbf{y}}{|b||d|\sqrt{\mathbf{x}'\mathbf{C}\mathbf{x} \cdot \mathbf{y}'\mathbf{C}\mathbf{y}}} = \frac{bd}{|bd|}r_{xy}\,.$$ □

3.3. Consider the 3×3 data matrix

$$\mathbf{U} = (\mathbf{1}_3 : \mathbf{x} : \mathbf{y}) = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}.$$

Confirm that the area of the triangle with vertices (x_i, y_i), $i = 1, 2, 3$, is $\frac{1}{2}|\det(\mathbf{U})|^*$. Show that $\det(\mathbf{U}) = 0 \iff$ the three data points lie in the same line, and hence, assuming that x and y have nonzero variances, we have $r_{xy}^2 = 1 \iff \det(\mathbf{U}) = 0$.

* Notice: in the book we have erroneously $\frac{1}{2}\det(\mathbf{U})$.

• Solution to Ex. 3.3:

Subtracting the first row from the second and third row shows that

$$|\mathbf{U}| = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x_3 - x_1 & y_3 - y_1 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$
$$= (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\,. \tag{1}$$

Consider a triangle OFG, say, in $\mathbb{R}^2$, formed by the points $\mathbf{0} = \binom{0}{0}$, $\mathbf{f} = \binom{a_1}{b_1}$, and $\mathbf{g} = \binom{a_2}{b_2}$. We will next show that

$$\text{area}(OFG) = \tfrac{1}{2}\,|\det(\mathbf{L})| = \tfrac{1}{2}\,|a_1 b_2 - a_2 b_1|\,, \tag{2}$$

where $\mathbf{L} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}' \\ \mathbf{g}' \end{pmatrix}$. Proceeding as in Section 5.7, we get

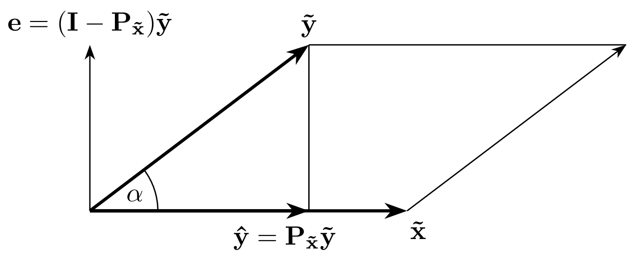

Figure 3.6 This is Figure 5.1. Replace here $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with $\mathbf{f}$ and $\mathbf{g}$, respectively.

$$\text{area}^2(OFG) = \tfrac{1}{4}\,\|\mathbf{f}\|^2\|\mathbf{e}\|^2,$$

where $\mathbf{e} = (\mathbf{I}_2 - \mathbf{P}_{\mathbf{f}})\mathbf{g}$. Now we have

$$\|\mathbf{e}\|^2 = \mathbf{g}'(\mathbf{I}_2 - \mathbf{P_f})\mathbf{g} = \mathbf{g}'[\mathbf{I}_2 - \mathbf{f}(\mathbf{f}'\mathbf{f})^{-1}\mathbf{f}']\mathbf{g}$$
$$= \mathbf{g}'\mathbf{g} - \frac{(\mathbf{f}'\mathbf{g})^2}{\mathbf{f}'\mathbf{f}} = \mathbf{g}'\mathbf{g}[1 - \cos^2(\mathbf{f},\mathbf{g})]\,,$$

and hence

$$\begin{aligned}\text{area}^2(OFG) &= \tfrac{1}{4}\,\mathbf{f}'\mathbf{f}\Big[\mathbf{g}'\mathbf{g} - \frac{(\mathbf{f}'\mathbf{g})^2}{\mathbf{f}'\mathbf{f}}\Big] \\ &= \tfrac{1}{4}\,[\mathbf{f}'\mathbf{f}\cdot\mathbf{g}'\mathbf{g} - (\mathbf{f}'\mathbf{g})^2] \\ &= \tfrac{1}{4}\,|\mathbf{L}\mathbf{L}'| = \tfrac{1}{4}\,|\mathbf{L}|^2,\end{aligned}$$

where $\mathbf{L} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}' \\ \mathbf{g}' \end{pmatrix}$. Thus (2) is proved.

Let's go now back to the original problem, i.e., finding the area of the triangle TUV, say, in $\mathbb{R}^2$, formed by the points $\mathbf{t} = \binom{x_1}{y_1}$, $\mathbf{u} = \binom{x_2}{y_2}$, and $\mathbf{v} = \binom{x_3}{y_3}$. We can move this triangle to the origin by subtracting $\mathbf{t}$ from $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{u}_0 = \mathbf{u} - \mathbf{t}$, $\mathbf{v}_0 = \mathbf{v} - \mathbf{t}$, and thus we obtain the triangle OU_0V_0, whose area is obviously the absolute value of

$$\frac{1}{2}\begin{vmatrix} \mathbf{u}' - \mathbf{t}' \\ \mathbf{v}' - \mathbf{t}' \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}.$$

Comment: Very likely there exist proofs simpler than that above.

Consider then the last claim: Show that $\det(\mathbf{U}) = 0 \iff$ the three data points lie in the same line, and hence, assuming that x and y have nonzero variances, we have $r_{xy}^2 = 1 \iff \det(\mathbf{U}) = 0$.

Obviously, in view of the nonzero variances, $\det(\mathbf{U}) = 0 \iff \text{rk}(\mathbf{U}) = 2$. Now $\text{rk}(\mathbf{U}) = 2$ implies that $\mathbf{y} = a\mathbf{1} + b\mathbf{x}$ for some a and b ($b \neq 0$), i.e., $y_i = a + bx_i$, $i = 1, 2, 3$, and hence $r_{xy}^2 = 1$. □

3.4. Consider the variables x_1 and x_2 and their sum $y = x_1 + x_2$. Let the corresponding variable vectors be $\mathbf{y}$, $\mathbf{x}_1$, and $\mathbf{x}_2$. Assume that $\mathbf{x}_1$ and $\mathbf{x}_2$ are centered and of equal length so that $\mathbf{x}_1'\mathbf{x}_1 = \mathbf{x}_2'\mathbf{x}_2 = d^2$. Show that

$$\text{var}_\text{s}(y) = \tfrac{1}{n-1}2d^2(1 + r_{12})\,, \quad \text{cor}_\text{s}^2(x_1, y) = \tfrac{1}{2}(1 + r_{12})\,.$$

Moreover, let $\mathbf{u} = \mathbf{x}_1 - \mathbf{P_y}\mathbf{x}_1$ and $\mathbf{v} = \mathbf{x}_2 - \mathbf{P_y}\mathbf{x}_2$. Show that $\mathbf{u} = -\mathbf{v}$ and thereby $r_{12\cdot y} = \text{cor}_\text{d}(\mathbf{u}, \mathbf{v}) = -1$.

• Solution to Ex. 3.4:

In view of $\mathbf{x}_1'\mathbf{x}_2 = d^2 r_{12}$, we have

$$\begin{aligned}
\mathrm{SS}_y &= \mathbf{y}'\mathbf{y} = \mathbf{x}_1'\mathbf{x}_1 + \mathbf{x}_2'\mathbf{x}_2 + 2\mathbf{x}_1'\mathbf{x}_2 \\
&= 2d^2 + 2d^2 r_{12} = 2d^2(1 + r_{12}) \,, \\
\mathrm{SP}_{x_1 y} &= \mathbf{x}_1'\mathbf{y} = \mathbf{x}_1'(\mathbf{x}_1 + \mathbf{x}_2) = d^2 + d^2\, r_{12} = d^2(1 + r_{12}) \,, \\
\mathrm{cor}_{\mathrm{s}}^2(x_1, y) &= \frac{[d^2(1 + r_{12})]^2}{d^2 \cdot 2d^2(1 + r_{12})} = \tfrac{1}{2}(1 + r_{12}) \,.
\end{aligned}$$

We observe that $\mathbf{u} = -\mathbf{v}$ because

$$\begin{aligned}
\mathbf{u} &= (\mathbf{I} - \mathbf{P}_{\mathbf{y}})\mathbf{x}_1 = (\mathbf{I} - \mathbf{P}_{\mathbf{y}})(\mathbf{y} - \mathbf{x}_2) \\
&= (\mathbf{I} - \mathbf{P}_{\mathbf{y}})\mathbf{y} - (\mathbf{I} - \mathbf{P}_{\mathbf{y}})\mathbf{x}_2 \\
&= -(\mathbf{I} - \mathbf{P}_{\mathbf{y}})\mathbf{x}_2 = -\mathbf{v} \,.
\end{aligned}$$

3.5. Let the variable vectors $\mathbf{x}$ and $\mathbf{y}$ be centered and of equal length.

(a) What is $\mathrm{cor}_{\mathrm{s}}(x - y, x + y)$?
(b) Draw a figure from which you can conclude the result above.
(c) What about if $\mathbf{x}'\mathbf{x} = \mathbf{y}'\mathbf{y} + 0.001$?

• Solution to Ex. 3.5:

Denoting $u = x + y, v = x - y,\ \mathbf{x}'\mathbf{x} = a^2,\ \mathbf{y}'\mathbf{y} = b^2$, we have

$$\begin{aligned}
\mathrm{SP}_{uv} &= (\mathbf{x} + \mathbf{y})'(\mathbf{x} - \mathbf{y}) = \mathbf{x}'\mathbf{x} - \mathbf{y}'\mathbf{y} = a^2 - b^2 \,, \\
\mathrm{SS}_v &= (\mathbf{x} - \mathbf{y})'(\mathbf{x} - \mathbf{y}) = \mathbf{x}'\mathbf{x} + \mathbf{y}'\mathbf{y} - 2\mathbf{x}'\mathbf{y} = a^2 + b^2 - 2\mathbf{x}'\mathbf{y} \,, \\
\mathrm{SS}_u &= (\mathbf{x} + \mathbf{y})'(\mathbf{x} + \mathbf{y}) = \mathbf{x}'\mathbf{x} + \mathbf{y}'\mathbf{y} + 2\mathbf{x}'\mathbf{y} = a^2 + b^2 + 2\mathbf{x}'\mathbf{y} \,, \\
r_{uv} &= \frac{a^2 - b^2}{\sqrt{(a^2 + b^2 - 2\mathbf{x}'\mathbf{y})(a^2 + b^2 + 2\mathbf{x}'\mathbf{y})}} \,.
\end{aligned}$$

3.6. Consider the variable vectors $\mathbf{x}$ and $\mathbf{y}$ which are centered and of unit length. Let $\mathbf{u}$ be the residual when $\mathbf{y}$ is explained by $\mathbf{x}$ and $\mathbf{v}$ be the residual when $\mathbf{x}$ is explained by $\mathbf{y}$. What is the $\mathrm{cor}_{\mathrm{d}}(\mathbf{u}, \mathbf{v})$? Draw a figure about the situation.

• Solution to Ex. 3.6:

$$\begin{aligned}\mathbf{u} &= (\mathbf{I} - \mathbf{P_y})\mathbf{x} = (\mathbf{I} - \mathbf{yy}')\mathbf{x} = \mathbf{x} - r_{xy}\mathbf{y}\,,\\ \mathbf{v} &= (\mathbf{I} - \mathbf{P_x})\mathbf{y} = (\mathbf{I} - \mathbf{xx}')\mathbf{y} = \mathbf{y} - r_{xy}\mathbf{x}\,,\\ \mathrm{SS}_u &= \mathbf{x}'(\mathbf{I} - \mathbf{yy}')\mathbf{x} = 1 - r_{xy}^2\,,\\ \mathrm{SS}_v &= \mathbf{y}'(\mathbf{I} - \mathbf{xx}')\mathbf{y} = 1 - r_{xy}^2\,,\\ \mathrm{SP}_{uv} &= \mathbf{x}'(\mathbf{I} - \mathbf{yy}')(\mathbf{I} - \mathbf{xx}')\mathbf{y}\\ &= (\mathbf{x} - r_{xy}\mathbf{y})'(\mathbf{y} - r_{xy}\mathbf{x})\\ &= r_{xy} - r_{xy} - r_{xy} + r_{xy}^3 = -r_{xy}(1 - r_{xy}^2)\,.\end{aligned}$$

Notice that $\mathbf{u}$ and $\mathbf{v}$ are centered since $\mathbf{x}$ and $\mathbf{y}$ are centered. □

3.7. Consider the variable vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ which are centered and of unit length and whose correlation matrix is

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & -1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 1 \end{pmatrix}.$$

Show that $\mathbf{x} - \mathbf{y} + \sqrt{2}\mathbf{z} = \mathbf{0}$.

• SOLUTION TO EX. 3.7:

Denoting $\mathbf{U} = (\mathbf{x} : \mathbf{y} : \mathbf{z})$ and $\mathbf{a} = (1, -1, \sqrt{2})'$ we have $\mathbf{R} = \mathbf{U}'\mathbf{U}$ and

$$\mathbf{U}'\mathbf{U}\mathbf{a} = \mathbf{R}\mathbf{a} = \begin{pmatrix} 1 & 0 & -1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now $\mathbf{U}'\mathbf{U}\mathbf{a} = \mathbf{0}$ implies $\mathbf{U}\mathbf{a} = \mathbf{0}$. □

3.8. Consider the 17-dimensional variable vectors $\mathbf{x} = (1, \dots, 1, 1, 4)'$ and $\mathbf{y} = (-1, \dots, -1, -1 + 0.001, 4)'$. Find $\mathrm{cor_d}(\mathbf{x}, \mathbf{y})$ and $\cos(\mathbf{x}, \mathbf{y})$.

• SOLUTION TO EX. 3.8:

$\mathbf{x}'\mathbf{x} = 16 + 16 = 32$, $\mathbf{y}'\mathbf{y} = 31 + 0.999^2$, $\mathbf{x}'\mathbf{y} = -16 + 0.001 + 16 = 0.001$, and hence

$$\cos(\mathbf{x}, \mathbf{y}) \approx \frac{0.001}{32} = \frac{1}{3200}.$$

Correlation $r_{xy} \approx 1$, because 16 observations lie on the line $y = \frac{5}{8}x - \frac{8}{3}$; the observation $(1, -0.999)$ is a bit out of this line. □

3.9. Suppose that $\mathbf{U} = (\mathbf{x} : \mathbf{y} : \mathbf{z}) \in \mathbb{R}^{100\times 3}$ is a data matrix, where the x-values represent the results of throwing a dice and y-values are observations from the normal distribution $\mathrm{N}(0,1)$, and $\mathbf{z} = \mathbf{x} + \mathbf{y}$. What is your guess for the length of the variable vector $\mathbf{z}$?

• SOLUTION TO EX. 3.9:

Theoretically, $\mu_x = 0,\ \mu_y = 3.5,\ \sigma_x^2 = 1,\ \sigma_y^2 = \frac{35}{12}$, and hence

$$\begin{aligned} 1 &\approx s_x^2 \approx \frac{1}{99}\mathbf{x}'\mathbf{x} \implies \mathbf{x}'\mathbf{x} \approx 99\,, \\ \frac{35}{12} &\approx s_y^2 \approx \frac{1}{99}(\mathbf{y}'\mathbf{y} - 100\cdot 3.5^2) \implies \mathbf{y}'\mathbf{y} \approx 99\cdot\frac{35}{12} + 100\cdot\frac{49}{4}\,, \\ \mathbf{z}'\mathbf{z} &\approx \mathbf{x}'\mathbf{x} + \mathbf{y}'\mathbf{y}\,. \end{aligned}$$

3.10. Consider a 3×3 centering matrix

$$\mathbf{C} = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}.$$

Find the following:

(a) a basis for $\mathscr{C}(\mathbf{C})$,
(b) orthonormal bases for $\mathscr{C}(\mathbf{C})$ and $\mathscr{C}(\mathbf{C})^\perp$,
(c) eigenvalues and orthonormal eigenvectors of $\mathbf{C}$,
(d) $\mathbf{C}^+$, and $\mathbf{C}^-$ which has (i) rank 2, (ii) rank 3.

• SOLUTION TO EX. 3.10:

(a) $\mathscr{C}(\mathbf{C}) = \{\mathbf{y} \in \mathbb{R}^3 : \exists\,\mathbf{x} \text{ such that } \mathbf{y} = \mathbf{C}\mathbf{x}\}$, i.e., $\mathscr{C}(\mathbf{C})$ is the set of all vectors which are centered or in other words, those vectors which satisfy $\mathbf{x}'\mathbf{1}_3 = 0$, so that $\mathscr{C}(\mathbf{C}) = \mathscr{C}(\mathbf{1}_3)^\perp$. The fact $\mathrm{rk}(\mathbf{C}) = 2$ can be concluded in many ways. For example, we always have

$$\mathrm{rk}(\mathbf{I}_n - \mathbf{P}_\mathbf{A}) = n - \mathrm{rk}(\mathbf{A})\,,$$

and so $\mathrm{rk}(\mathbf{C}) = \mathrm{rk}(\mathbf{I}_3 - \mathbf{P}_\mathbf{1}) = 3 - \mathrm{rk}(\mathbf{1}) = 2$. Any two columns of $\mathbf{C}$ create a basis for $\mathscr{C}(\mathbf{C})$ as well as the columns of

$$\mathbf{A}_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \\ -1 & -1 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) $\mathbf{B} = \begin{pmatrix} -2/\sqrt{6} & 0 \\ 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix} = (\mathbf{b}_1 : \mathbf{b}_2)$ is an orthonormal basis for $\mathscr{C}(\mathbf{C})$.

$\mathscr{C}(\mathbf{C})^\perp = \mathscr{C}(\mathbf{1}_3) \implies \frac{1}{\sqrt{3}}\mathbf{1}_3$ is an orthonormal basis for $\mathscr{C}(\mathbf{C})^\perp$.

(c) The eigenvalues of an idempotent matrix $\mathbf{P}$, say, are zeros and ones (why?) and the number of ones is $\operatorname{rk}(\mathbf{P})$. Hence $\operatorname{ch}(\mathbf{C}) = \{1, 1, 0\}$.

- $\mathbf{C1} = 0 \cdot \mathbf{1}$, and so $\mathbf{1}$ is an eigenvector of $\mathbf{C}$ w.r.t. eigenvalue 0, i.e., $(0, \mathbf{1})$ is an eigenpair for $\mathbf{C}$.
- $\mathbf{Ct} = 1 \cdot \mathbf{t}$ iff $\mathbf{t}$ is a centered vector. Hence any centered vector $\mathbf{t}$ is an eigenvector of $\mathbf{C}$ w.r.t. eigenvalue 1. For example, the columns of $\mathbf{B}$ in (b) are orthonormal eigenvectors of $\mathbf{C}$ w.r.t. eigenvalue 1. We then have the equation
$$\mathbf{CB} = \mathbf{B}\,. \tag{1}$$
Postmultiplying (1) by an orthogonal $\mathbf{Q}_{2\times 2}$ yields
$$\mathbf{CBQ} = \mathbf{BQ}\,,$$
which shows that the columns of $\mathbf{BQ}$ are also orthonormal eigenvectors of $\mathbf{C}$ w.r.t. eigenvalue 1. Recall that according to Theorem 18 (p. 357) the following holds for multiple eigenvalues:

- Consider the distinct eigenvalues of $\mathbf{A}$, $\lambda_{\{1\}} > \cdots > \lambda_{\{s\}}$, and let $\mathbf{T}_{\{i\}}$ be an $n \times m_i$ matrix consisting of the orthonormal eigenvectors corresponding to $\lambda_{\{i\}}$; m_i is the multiplicity of $\lambda_{\{i\}}$. With this ordering, $\mathbf{\Lambda}$ is unique and $\mathbf{T}$ is unique up to postmultiplying by a blockdiagonal matrix $\mathbf{U} = \operatorname{blockdiag}(\mathbf{U}_1, \dots, \mathbf{U}_s)$, where $\mathbf{U}_i$ is an orthogonal $m_i \times m_i$ matrix.

(d) $\mathbf{C} = \mathbf{C}^+$, which is easy to establish; $\operatorname{rk}(\mathbf{C}) = 2$, and so $\mathbf{C}$ is a generalized inverse $\mathbf{C}^-$ which has rank 2.

Denote
$$\mathbf{T} = (\mathbf{t}_1 : \mathbf{t}_2 : \mathbf{t}_3) = (\mathbf{T}_1 : \tfrac{1}{\sqrt{3}}\mathbf{1}_3)\,,$$
where $\mathbf{T}_1 = \mathbf{B}$, as in (b). Then the vectors $\mathbf{t}_i$ are the orthonormal eigenvectors of $\mathbf{C}$ and $\mathbf{C}$ has the eigenvalue decomposition
$$\mathbf{C} = \mathbf{T\Lambda T}' = \mathbf{T}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{T}' = \mathbf{T}_1\mathbf{T}_1' = \mathbf{t}_1\mathbf{t}_1' + \mathbf{t}_2\mathbf{t}_2'\,.$$

In light of Section 19.5 (pp. 407–408),
$$\mathbf{T}\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ c & d & e \end{pmatrix}\mathbf{T}' \in \{\mathbf{C}^-\} \quad \text{for all } a, b, c, d, e \in \mathbb{R}\,. \tag{2}$$

In particular,

$$\mathbf{G} = \mathbf{T}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\alpha\end{pmatrix}\mathbf{T}' = \mathbf{t}_1\mathbf{t}_1' + \mathbf{t}_2\mathbf{t}_2' + \alpha\mathbf{1}_3\mathbf{1}_3' \in \{\mathbf{C}^-\}\,,$$

i.e.,

$$\mathbf{G} = \mathbf{t}_1\mathbf{t}_1' + \mathbf{t}_2\mathbf{t}_2' + \alpha\mathbf{1}_3\mathbf{1}_3' = \mathbf{C} + \alpha\mathbf{1}_3\mathbf{1}_3' \in \{\mathbf{C}^-\} \quad \text{for all } \alpha \in \mathbb{R}\,.$$

If $\alpha \neq 0$ then $\operatorname{rk}(\mathbf{G}) = 3$, and $\mathbf{G}$ is a positive definite generalized inverse of $\mathbf{C}$. Of course there are many other positive definite generalized inverses for $\mathbf{C}$.

Let us next consider the conditions under which $\mathbf{G}$ in (2) is a symmetric nonnegative definite generalized inverse of $\mathbf{C}$; denote this set as

$$\{\mathbf{C}^-_{\mathrm{nnd}}\}\,.$$

We observe that

$$\mathbf{G} = \mathbf{T}\begin{pmatrix}1 & 0 & f_1\\ 0 & 1 & f_2\\ f_1 & f_2 & \delta\end{pmatrix}\mathbf{T}' = \mathbf{T}\begin{pmatrix}\mathbf{I}_2 & \mathbf{f}\\ \mathbf{f}' & \delta\end{pmatrix}\mathbf{T}' \in \mathrm{NND}_3$$

if and only if (why?)

$$\mathbf{K} := \begin{pmatrix}\mathbf{I}_2 & \mathbf{f}\\ \mathbf{f}' & \delta\end{pmatrix} \in \mathrm{NND}_3\,,$$

which in view of (14.7) and (14.8) (p. 306) holds if and only if

$$\mathbf{f}'\mathbf{f} \leq \delta\,. \tag{3}$$

For notational convenience, we can replace $\mathbf{K}$ with

$$\mathbf{L} = \begin{pmatrix}\mathbf{I}_2 & \sqrt{3}\,\mathbf{f}\\ \sqrt{3}\,\mathbf{f}' & 3\delta\end{pmatrix},$$

which is nonnegative definite iff (3) holds.

Two side-questions:
(a) Is the symmetry of

$$\begin{pmatrix}1 & 0 & a\\ 0 & 1 & b\\ c & d & e\end{pmatrix} := \mathbf{N}$$

necessary for the symmetry of $\mathbf{TNT}'$? (Yes.)
(b) Is the nonnegative definiteness of $\mathbf{K}$ necessary for the nonnegative definiteness of $\mathbf{TKT}'$? (Yes.)

Let's calculate the matrix $\mathbf{G} = \mathbf{TLT}'$:

$$\begin{aligned}\mathbf{G} = \mathbf{TLT}' &= \mathbf{T}\begin{pmatrix} \mathbf{I}_2 & \sqrt{3}\,\mathbf{f} \\ \sqrt{3}\,\mathbf{f}' & 3\delta \end{pmatrix}\mathbf{T}' \\ &= \mathbf{T}_1\mathbf{T}_1' + \mathbf{t}_3\sqrt{3}\,\mathbf{f}'\mathbf{T}_1' + \mathbf{T}_1\sqrt{3}\,\mathbf{f}\mathbf{t}_3' + \delta\, 3\,\mathbf{t}_3\mathbf{t}_3' \\ &= \mathbf{C} + \mathbf{1}\mathbf{f}'\mathbf{T}_1' + \mathbf{T}_1\mathbf{f}\mathbf{1}' + \delta\,\mathbf{1}\mathbf{1}'.\end{aligned}$$

Now any matrix of the form

$$\mathbf{G} = \mathbf{C} + \mathbf{1}\mathbf{f}'\mathbf{T}_1' + \mathbf{T}_1\mathbf{f}\mathbf{1}' + \delta\,\mathbf{1}\mathbf{1}' \tag{4}$$

is a nonnegative definite generalized inverse of $\mathbf{C}$ for any $\mathbf{f} \in \mathbb{R}^2$ and $\delta \in \mathbb{R}$ which satisfy

$$\mathbf{f}'\mathbf{f} \le \delta\,. \tag{5}$$

We stop here for a while and advise the reader to have a look at the Exercise 15.10 (p. 342) and the references therein.

After a short break, le'ts go back to business. According to Exercise 15.10 it *seems* that the set of nnd matrices given in (4) equals the set of nnd matrices of the form

$$\mathbf{V} = \mathbf{I}_3 + \mathbf{a}\mathbf{1}' + \mathbf{1}\mathbf{a}' := \mathbf{I}_3 + \mathbf{W}, \tag{6}$$

where $\mathbf{a} \in \mathbb{R}^3$ is an arbitrary vector subject to the condition that $\mathbf{W}$ is nonnegative definite. Actually in Exercise 15.10 we are dealing with positive definiteness and hence seems is *seems* above. When is $\mathbf{V}$ in (6) nnd? We observe that

$$\operatorname{ch}(\mathbf{V}) = \{1 + \operatorname{ch}(\mathbf{a}\mathbf{1}' + \mathbf{1}\mathbf{a}')\}\,. \tag{7}$$

Now consider the equation

$$\mathbf{W}\mathbf{q} = (\mathbf{a}\mathbf{1}' + \mathbf{1}\mathbf{a}')\mathbf{q} = 0\mathbf{q} = \mathbf{0}$$

i.e.,

$$\mathbf{a}(\mathbf{1}'\mathbf{q}) + \mathbf{1}(\mathbf{a}'\mathbf{q}) = \mathbf{0}\,.$$

If $\operatorname{rk}(\mathbf{1} : \mathbf{a}) = 2$, then $\mathbf{q} \in \mathscr{C}(\mathbf{1} : \mathbf{a})^\perp$ and thereby 0 is an eigenvalue of $\mathbf{W}$ of multiplicity $1 (= n - 2)$ with $\mathbf{q}$ being the corresponding eigenvector. The remaining two eigenvalues of $\mathbf{W} = \mathbf{a}\mathbf{1}' + \mathbf{1}\mathbf{a}'$ are $\mathbf{1}'\mathbf{a} \pm \sqrt{n\mathbf{a}'\mathbf{a}}$. How to prove this? We leave this open and refer to the references; in particular, Farebrother (1987, Cor. 1). In any event, $\mathbf{V}$ is nnd iff

$$1 + \mathbf{1}'\mathbf{a} - \sqrt{n\mathbf{a}'\mathbf{a}} \ge 0\,, \quad \text{i.e.,} \quad 1 \ge \sqrt{n\mathbf{a}'\mathbf{a}} - \mathbf{1}'\mathbf{a}\,. \tag{8}$$

If $\operatorname{rk}(\mathbf{1} : \mathbf{a}) = 1$, i.e., $\mathbf{a} = c\mathbf{1}$ for some nonzero $c \in \mathbb{R}$, then $\mathbf{V} = \mathbf{I}_3 + 2c\mathbf{1}\mathbf{1}'$ whose eigenvalues are $\{1 + \operatorname{ch}(2c\mathbf{1}\mathbf{1}')\} = \{1, 1, 6c\}$.

It's time to stop here. There are some related considerations in the solution of Exercise 15.10. We may mention that Chaganty & Vaish (1997, Cor. 2.1) characterized the class of all nnd generalized inverses of the centering matrix $\mathbf{C} \in \mathbb{R}^{n\times n}$ as

$$\mathbf{V} = \mathbf{C} + \mathbf{1}\mathbf{b}' + \mathbf{b}\mathbf{1}' - \bar{b}\mathbf{1}\mathbf{1}', \tag{9}$$

where $\mathbf{b} \in \mathbb{R}^n$ is such that

$$\mathbf{b}'\mathbf{C}\mathbf{b} \leq \bar{b}\,, \quad \text{and } \bar{b} = \mathbf{1}'\mathbf{b}/n\,.$$

Some references; those not appearing in the Tricks References, written in full:

Chaganty, N. Rao & Vaish, A. K. (1997). An invariance property of common statistical tests. *Linear Algebra and its Applications*, 264, 421–437.

Farebrother, R. W. (1987). Three theorems with applications to Euclidean distance matrices. *Linear Algebra and its Applications*, 95, 11–16.

Jensen, D. R. (1996). Structured dispersion and validity in linear inference. *Linear Algebra and its Applications*, 249, 189–196.

Jensen, D. R. & Srinivasan, S. S. (2004). Matrix equivalence classes with applications. *Linear Algebra and its Applications*, 388, 249–260.

Mathew (1985).

Sharpe, G. E. & Styan, G. P. H. (1965). Circuit duality and the general network inverse. *IEEE Trans. Circuit Theory CT-12*, 22–27.

Styan & Subak-Sharpe (1997). □

3.11. Suppose that the variable vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are centered and of unit length. Show that corresponding to (3.8) (p. 93),

$$r_{xy\cdot z} = 0 \iff \mathbf{y} \in \mathscr{C}(\mathbf{Q_z x})^\perp = \mathscr{C}(\mathbf{x} : \mathbf{z})^\perp \boxplus \mathscr{C}(\mathbf{z})\,,$$

where $\mathbf{Q_z} = \mathbf{I}_n - \mathbf{P_z}$. See also Exercise 8.7 (p. 185).