4.5 Exercises: Some Solutions (November 3, 2011)

4.1. Find all generalized inverses of $\mathbf{A} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. What is $\mathbf{A}^+$? What about the reflexive generalized inverse of $\mathbf{A}$? Characterize the set of symmetric nonnegative definite generalized inverses of $\mathbf{A}$.

• Solution to Ex. 4.1:

Writing

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we see that any $\mathbf{G} = \begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix} \in \{\mathbf{A}^-\}$ for all α, β, γ. Let us then consider the generalized inverses via the SVD of $\mathbf{A}$.

$|\mathbf{A} - \lambda\mathbf{I}_2| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 0-\lambda \end{vmatrix} = -\lambda(1-\lambda) = 0$

$\implies \operatorname{ch}(\mathbf{A}) = \{1, 0\} = \{\lambda_1, \lambda_2\}.$

$\mathbf{A}\mathbf{t}_1 = \mathbf{t}_1 \implies \begin{pmatrix} t_{11} \\ 0 \end{pmatrix} = \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix} \implies \mathbf{t}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$

$\mathbf{A}\mathbf{t}_2 = 0\mathbf{t}_2 \implies \begin{pmatrix} t_{21} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \mathbf{t}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$

EVD (= SVD) of $\mathbf{A}$ is extremely simple: $\mathbf{A} = \mathbf{I}_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{I}_2$.

Recall that according to Section 19.5 (pp. 407–408) the following holds:

If $\mathbf{A}_{n\times m}$ has a singular value decomposition $\mathbf{A} = \mathbf{U} \begin{pmatrix} \mathbf{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}'$, then

$$\mathbf{G} \in \{\mathbf{A}^-\} \iff \mathbf{G} = \mathbf{V} \begin{pmatrix} \mathbf{\Delta}_1^{-1} & \mathbf{K} \\ \mathbf{L} & \mathbf{N} \end{pmatrix} \mathbf{U}',$$

$$\mathbf{G} \in \{\mathbf{A}_{12}^-\} \iff \mathbf{G} = \mathbf{V} \begin{pmatrix} \mathbf{\Delta}_1^{-1} & \mathbf{K} \\ \mathbf{L} & \mathbf{L}\mathbf{\Delta}_1\mathbf{K} \end{pmatrix} \mathbf{U}',$$

$$\mathbf{G} = \mathbf{A}^+ \iff \mathbf{G} = \mathbf{V} \begin{pmatrix} \mathbf{\Delta}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}',$$

where $\mathbf{K}$, $\mathbf{L}$, and $\mathbf{N}$ are arbitrary matrices. Hence for $\mathbf{A} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ we have the following:

- $\mathbf{G} \in \{\mathbf{A}_{12}^{-}\} \iff \mathbf{G} = \begin{pmatrix} 1 & \alpha \\ \beta & \alpha\beta \end{pmatrix}$ and such $\mathbf{G}$ is symmetric when $\mathbf{G} = \begin{pmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{pmatrix}$.
- $\mathbf{A}^{+} = \mathbf{A}$.
- $\mathbf{G}$ is a symmetric nonnegative definite generalized inverse of $\mathbf{A}$ if and only if $\mathbf{G} = \begin{pmatrix} 1 & \alpha \\ \alpha & \gamma^2 \end{pmatrix}$, where $\gamma^2 \geq \alpha^2$. (Why?) □

4.2. Denote

$$\mathbf{A}_{n\times m} = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{pmatrix}, \quad \mathbf{G}_{m\times n} = \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\operatorname{rank}(\mathbf{B}_{r\times r}) = r = \operatorname{rank}(\mathbf{A})$. Show that $\mathbf{G} \in \{\mathbf{A}^{-}\}$.

• Solution to Ex. 4.2:

$\mathbf{AGA} = \mathbf{A}$. □

4.3. Show that $(\mathbf{A}'\mathbf{A})^{-}\mathbf{A}'$ is a $\{123\}$-inverse of $\mathbf{A}$.

• Solution to Ex. 4.3:

$\mathbf{A} \cdot (\mathbf{A}'\mathbf{A})^{-}\mathbf{A}' \cdot \mathbf{A} = \mathbf{A}$,
$(\mathbf{A}'\mathbf{A})^{-}\mathbf{A}' \cdot \mathbf{A} \cdot (\mathbf{A}'\mathbf{A})^{-}\mathbf{A}' = (\mathbf{A}'\mathbf{A})^{-}\mathbf{A}'$,
$[\mathbf{A} \cdot (\mathbf{A}'\mathbf{A})^{-}\mathbf{A}']' = \mathbf{A} \cdot (\mathbf{A}'\mathbf{A})^{-}\mathbf{A}$.
The above equalities follow easily from the fact $\mathbf{A}(\mathbf{A}'\mathbf{A})^{-}\mathbf{A} = \mathbf{P}_{\mathbf{A}}$. □

4.4. Prove: $\mathbf{AGA} = \mathbf{A}$ & $\operatorname{rank}(\mathbf{G}) = \operatorname{rank}(\mathbf{A}) \implies \mathbf{G} \in \{\mathbf{A}_{12}^{-}\}$, i.e., $\mathbf{G}$ is a reflexive generalized inverse of $\mathbf{A}$.

• Solution to Ex. 4.4:

$\mathbf{G} \in \{\mathbf{A}_{12}^{-}\} \iff \mathbf{AGA} = \mathbf{A}$ and $\mathbf{GAG} = \mathbf{G}$.
Suppose that $\mathbf{AGA} = \mathbf{A}$ & $\operatorname{rk}(\mathbf{G}) = \operatorname{rk}(\mathbf{A})$. Then

$$\mathbf{AGAG} = \mathbf{AG}\,.$$

Because always $\operatorname{rk}(\mathbf{AG}) = \operatorname{rk}(\mathbf{A})$, and by assumption $\operatorname{rk}(\mathbf{A}) = \operatorname{rk}(\mathbf{G})$, we can cancel $\mathbf{A}$ on the left on both sides of the above equality thus yielding $\mathbf{GAG} = \mathbf{AG}$. □

4.5. Confirm:

(a) $\begin{pmatrix} \mathbf{0}_{p\times n} \\ \mathbf{A}^- \end{pmatrix} \in \{(\mathbf{0}_{n\times p} : \mathbf{A}_{n\times m})^-\}$,

(b) $\det(\mathbf{P}) \neq 0, \det(\mathbf{Q}) \neq 0 \implies \mathbf{Q}^{-1}\mathbf{A}^-\mathbf{P}^{-1} \in \{(\mathbf{PAQ})^-\}, \mathbf{Q}^{-1}\mathbf{A}^+\mathbf{P}^{-1} = (\mathbf{PAQ})^+$.

• Solution to Ex. 4.5:

$$\text{(a)} \quad (\mathbf{0}_{n\times p} : \mathbf{A}_{n\times m}) \begin{pmatrix} \mathbf{0}_{p\times n} \\ \mathbf{A}^- \end{pmatrix} (\mathbf{0}_{n\times p} : \mathbf{A}_{n\times m}) = \mathbf{AA}^-(\mathbf{0}_{n\times p} : \mathbf{A}_{n\times m}) = (\mathbf{0}_{n\times p} : \mathbf{A}_{n\times m})\,.$$

(b) In view of

$$\mathbf{PAQ} \cdot \mathbf{Q}^{-1}\mathbf{A}^-\mathbf{P}^{-1} \cdot \mathbf{PAQ} = \mathbf{PAA}^-\mathbf{AQ} = \mathbf{PAQ}\,,$$

we have $\mathbf{Q}^{-1}\mathbf{A}^-\mathbf{P}^{-1} \in \{(\mathbf{PAQ})^-\}$.

(mp1) $\mathbf{PAQ} \cdot \mathbf{Q}^{-1}\mathbf{A}^+\mathbf{P}^{-1} \cdot \mathbf{PAQ} = \mathbf{PAA}^+\mathbf{AQ} = \mathbf{PAQ}\,,$

(mp2) $\mathbf{Q}^{-1}\mathbf{A}^+\mathbf{P}^{-1} \cdot \mathbf{PAQ} \cdot \mathbf{Q}^{-1}\mathbf{A}^+\mathbf{P}^{-1} = \mathbf{Q}^{-1}\mathbf{A}^+\mathbf{P}^{-1},$

(mp3) $\mathbf{PAQ} \cdot \mathbf{Q}^{-1}\mathbf{A}^+\mathbf{P}^{-1} = \mathbf{PAA}^+\mathbf{P}^{-1} = \mathbf{PP}_{\mathbf{A}}\mathbf{P}^{-1} = \text{ symmetric},$

(mp4) $\mathbf{Q}^{-1}\mathbf{A}^+\mathbf{P}^{-1} \cdot \mathbf{PAQ} = \mathbf{Q}^{-1}\mathbf{A}^+\mathbf{AQ} = \mathbf{Q}^{-1}\mathbf{P}_{\mathbf{A}'}\mathbf{Q} = \text{ symmetric}\,.$

However, the last two statements do no necessarily hold. Hence the text in the beginning of the exercise should be rephrased as: "Confirm or deny:". Notice that following holds:

$$\mathbf{P} \text{ and } \mathbf{Q} \text{ are orthogonal} \implies (\mathbf{PAQ})^+ = \mathbf{Q}'\mathbf{A}^+\mathbf{P}'.$$

□