5.14 Exercises: Some Solutions (November 10, 2011)

5.1. Show that for conformable matrices we have

$$\operatorname{rank}\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{pmatrix} = \operatorname{rank}(\mathbf{B}) + \operatorname{rank}(\mathbf{C}) + \operatorname{rank}\big[(\mathbf{I}-\mathbf{P}_{\mathbf{B}})\mathbf{A}(\mathbf{I}-\mathbf{P}_{\mathbf{C}'})\big]\,.$$

Marsaglia & Styan (1974a, Th. 19).

• SOLUTION TO EX. 5.1:

Consider $\mathbf{A}_{n\times m}$, $\mathbf{B}_{n\times p}$, $\mathbf{C}_{q\times m}$, and denote $\mathbf{Q}_{\mathbf{B}} = \mathbf{I}_n - \mathbf{P}_{\mathbf{B}}$. Then

$$\begin{aligned}
\operatorname{rk}\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{pmatrix} &= \operatorname{rk}\begin{pmatrix}\mathbf{B}\\ \mathbf{0}\end{pmatrix} + \operatorname{rk}\left((\mathbf{I}_{n+q} - \mathbf{P}_{\left(\begin{smallmatrix}\mathbf{B}\\ \mathbf{0}\end{smallmatrix}\right)})\begin{pmatrix}\mathbf{A}\\ \mathbf{C}\end{pmatrix}\right)\\
&= \operatorname{rk}(\mathbf{B}) + \operatorname{rk}\left(\begin{pmatrix}\mathbf{Q}_{\mathbf{B}} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_q\end{pmatrix}\begin{pmatrix}\mathbf{A}\\ \mathbf{C}\end{pmatrix}\right)\\
&= \operatorname{rk}(\mathbf{B}) + \operatorname{rk}\begin{pmatrix}\mathbf{Q}_{\mathbf{B}}\mathbf{A}\\ \mathbf{C}\end{pmatrix}\\
&= \operatorname{rk}(\mathbf{B}) + \operatorname{rk}(\mathbf{C}' : \mathbf{A}'\mathbf{Q}_{\mathbf{B}})\\
&= \operatorname{rk}(\mathbf{B}) + [\operatorname{rk}(\mathbf{C}') + \operatorname{rk}(\mathbf{Q}_{\mathbf{C}'}\mathbf{A}'\mathbf{Q}_{\mathbf{B}})]\\
&= \operatorname{rk}(\mathbf{B}) + \operatorname{rk}(\mathbf{C}) + \operatorname{rk}(\mathbf{Q}_{\mathbf{B}}\mathbf{A}\mathbf{Q}_{\mathbf{C}'})\,.
\end{aligned}$$

Notice that

$$\mathbf{P}_{\left(\begin{smallmatrix}\mathbf{B}\\ \mathbf{0}\end{smallmatrix}\right)} = \begin{pmatrix}\mathbf{B}\\ \mathbf{0}\end{pmatrix}(\mathbf{B}'\mathbf{B})^{-}(\mathbf{B}' : \mathbf{0}) = \begin{pmatrix}\mathbf{P}_{\mathbf{B}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}.$$

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