10.14 Exercises: Some Solutions (November 25, 2012)

Baksalary & Kala (1981a, p. 913) and Drygas (1983) showed that a linear statistic $\mathbf{Fy}$ is linearly sufficient for $\mathbf{X}\boldsymbol{\beta}$ under the model $\mathscr{M}$ if and only if the column space inclusion $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{WF}')$ holds; here $\mathbf{W} = \mathbf{V} + \mathbf{XLL}'\mathbf{X}'$ with $\mathbf{L}$ being an arbitrary matrix such that $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})$. The hard-working reader may prove the following proposition.

Proposition 10.12. (Page 257.) Let $\mathbf{W} = \mathbf{V} + \mathbf{XLL}'\mathbf{X}'$ be an arbitrary matrix satisfying $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})$. Then $\mathbf{Fy}$ is linearly sufficient for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V}\}$ if and only if any of the following equivalent statements holds:

(a) $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{WF}')$,

(b) $\mathscr{N}(\mathbf{F}) \cap \mathscr{C}(\mathbf{X} : \mathbf{V}) \subset \mathscr{C}(\mathbf{VX}^{\perp})$,

(c) $\operatorname{rank}(\mathbf{X} : \mathbf{VF}') = \operatorname{rank}(\mathbf{WF}')$,

(d) $\mathscr{C}(\mathbf{X}'\mathbf{F}') = \mathscr{C}(\mathbf{X}')$ and $\mathscr{C}(\mathbf{FX}) \cap \mathscr{C}(\mathbf{FVX}^{\perp}) = \{\mathbf{0}\}$,

(e) the best linear predictor of $\mathbf{y}$ based on $\mathbf{Fy}$, $\operatorname{BLP}(\mathbf{y}; \mathbf{Fy})$, is almost surely equal to a linear function of $\mathbf{Fy}$ which does not depend on $\boldsymbol{\beta}$.

Moreover, let $\mathbf{Fy}$ be linearly sufficient for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V}\}$. Then each BLUE of $\mathbf{X}\boldsymbol{\beta}$ under the transformed model $\{\mathbf{Fy}, \mathbf{FX}\boldsymbol{\beta}, \sigma^2\mathbf{FVF}'\}$ is the BLUE of $\mathbf{X}\boldsymbol{\beta}$ under the original model $\mathscr{M}$ and vice versa.

• Proof of Proposition 10.12:

Denote $\mathbf{U} = \mathbf{LL}'$ so that $\mathbf{W} = \mathbf{V} + \mathbf{XUX}'$. Following Baksalary & Kala (1981a, p. 914), let us begin by writing up the following lemma:

Lemma A. Let $\mathbf{K}\boldsymbol{\beta}$ be estimable under $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$. Then $\mathbf{Ay}$ is a BLUE of $\mathbf{K}\boldsymbol{\beta}$ if and only if

$$\mathbf{AW} = \mathbf{K}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}', \tag{1}$$

where $\mathbf{W}$ and $\mathbf{L}$ are defined so that

$$\mathbf{W} = \mathbf{V} + \mathbf{XLL}'\mathbf{X}', \quad \mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{W}). \tag{2}$$

To prove Lemma A, we know by Theorem 10 (p. 217) that $\mathbf{Ay}$ is a BLUE for estimable $\mathbf{K}\boldsymbol{\beta}$ if and only if

$$\mathbf{A}(\mathbf{X} : \mathbf{VM}) = (\mathbf{K} : \mathbf{0}), \quad \text{where } \mathbf{M} = \mathbf{I}_n - \mathbf{H}. \tag{3}$$

The general expression for $\mathbf{A}$ satisfying (3) is, see Prop. 10.5 (p. 229),

$$\mathbf{A} = \mathbf{K}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^- + \mathbf{N}(\mathbf{I}_n - \mathbf{P}_{\mathbf{W}}), \tag{4}$$

where $\mathbf{N}$ is free to vary. Postmultipying (4) by $\mathbf{W}$ gives

$$\mathbf{AW} = \mathbf{K}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\mathbf{W} = \mathbf{K}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}',$$

where we have used the property $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{W}')$; see Prop. 12.1 (p. 286). On the other hand, the general solution for $\mathbf{A}$ in (1) is precisely of the form (4) and thereby (1) implies that $\mathbf{Ay}$ is a BLUE for $\mathbf{K}\boldsymbol{\beta}$.

Baksalary & Kala (1981a, p. 914) formulate their result as follows:

Theorem A. Let $\mathbf{F}$ be a $q \times n$ matrix.

(i) A BLUE of $\mathbf{X}\boldsymbol{\beta}$ is obtainable as a linear function of $\mathbf{Fy}$ [this phraseology is later called *linear sufficiency*] if and only if

$$\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{WF}'), \tag{5}$$

or, equivalently,

$$\operatorname{rk}(\mathbf{X} : \mathbf{VF}') = \operatorname{rk}(\mathbf{WF}'). \tag{6}$$

(ii) If the condition of (i) is satisfied, then each BLUE of $\mathbf{X}\boldsymbol{\beta}$ in the transformed model $\mathscr{M}_F = \{\mathbf{Fy}, \mathbf{FX}\boldsymbol{\beta}, \mathbf{FVF}'\}$ is also a BLUE of $\mathbf{X}\boldsymbol{\beta}$ in the original model $\{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \mathbf{V}\}$, and vice versa.

To prove (i), we observe that in view of Lemma A, a BLUE of $\mathbf{X}\boldsymbol{\beta}$ is expressible as $\mathbf{CFy}$, for some $q \times n$ matrix $\mathbf{C}$, if and only if

$$\mathbf{CFW} = \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'. \tag{7}$$

The above equation has a solution for $\mathbf{C}$ if and only if

$$\mathscr{C}\big[\mathbf{X}[(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-]'\mathbf{X}'\big] \subset \mathscr{C}(\mathbf{WF}'). \tag{8}$$

But in view of Prop. 12.1 (p. 286) we have

$$\mathbf{X}'\mathbf{W}^-\mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}' = \mathbf{X}'$$

and thereby

$$\mathbf{X}[(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-]'\mathbf{X}'(\mathbf{W}^-)'\mathbf{X} = \mathbf{X}\,, \tag{9}$$

for any choices of the generalized inverses involved, which implies

$$\begin{aligned}\mathscr{C}(\mathbf{X}) &= \mathscr{C}\{\mathbf{X}[(\mathbf{X}'\mathbf{W}^-]'\mathbf{X})^-\mathbf{X}'(\mathbf{W}^-)'\mathbf{X}\} \\ &\subset \mathscr{C}\{\mathbf{X}[(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-]'\mathbf{X}'\} \subset \mathscr{C}(\mathbf{X}),\end{aligned}$$

and so

$$\mathscr{C}\big\{\mathbf{X}[(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-]'\mathbf{X}'\big]\big\} = \mathscr{C}(\mathbf{X})\,. \tag{10}$$

Note: In the above proof, the nonnegative definiteness of $\mathbf{W}$ does not seem to be needed!

To confirm the equivalence between (5) and (6), we first note that obviously (5) is equivalent to

$$\operatorname{rk}(\mathbf{X} : \mathbf{WF}') = \operatorname{rk}(\mathbf{WF}')\,,$$

but on the other hand, cf. (5.2) in Theorem 5 (p. 121),

$$\begin{aligned}\operatorname{rk}(\mathbf{X} : \mathbf{WF}') &= \operatorname{rk}(\mathbf{X}) + \operatorname{rk}[(\mathbf{I}_n - \mathbf{P}_{\mathbf{X}})\mathbf{WF}'] \\ &= \operatorname{rk}(\mathbf{X}) + \operatorname{rk}[(\mathbf{I}_n - \mathbf{P}_{\mathbf{X}})\mathbf{VF}'] \\ &= \operatorname{rk}(\mathbf{X} : \mathbf{VF}')\,, \end{aligned} \tag{11}$$

and thus (5) $\iff$ (6) and the proof of part (i) is completed.

To prove part (ii), Baksalary & Kala (1981a, p. 915) observe first that (5) implies

$$\mathscr{C}[\mathbf{X}'(\mathbf{W}^-)'\mathbf{X}] \subset \mathscr{C}[\mathbf{X}'(\mathbf{W}^-)'\mathbf{WF}'], \tag{12}$$

but it seems to be a bit simpler to write

$$\mathscr{C}(\mathbf{X}'\mathbf{W}^-\mathbf{X}) \subset \mathscr{C}(\mathbf{X}'\mathbf{W}^-\mathbf{WF}'). \tag{13}$$

Now $\mathbf{X}'\mathbf{W}^-\mathbf{W} = \mathbf{X}'$ and $\mathscr{C}(\mathbf{X}'\mathbf{W}^-\mathbf{X}) = \mathscr{C}(\mathbf{X}')$, and therefore (13) reduces to

$$\mathscr{C}(\mathbf{X}') \subset \mathscr{C}(\mathbf{X}'\mathbf{F}')\,, \quad \text{i.e.,} \quad \mathscr{C}(\mathbf{X}') = \mathscr{C}(\mathbf{X}'\mathbf{F}'). \tag{14}$$

This shows that the functions of $\mathbf{X}\boldsymbol{\beta}$, which are obviously estimable in the original model $\mathscr{M}$, are also estimable in the transformed model $\mathscr{M}_F$ (please confirm!). In view of the Lemma A, a statistic $\mathbf{CFy}$ is a BLUE of $\mathbf{X}\boldsymbol{\beta}$ in the model $\mathscr{M}_F$ if and only if

$$\mathbf{CFWF}' = \mathbf{X}[\mathbf{X}'\mathbf{F}'(\mathbf{FWF}')^-\mathbf{FX}]^-\mathbf{X}'\mathbf{F}', \tag{15}$$

where we have chosen the "$\mathbf{W}$-matrix" in the model $\mathscr{M}_F$ as

$$\mathbf{W}_F = \mathbf{F}(\mathbf{V} + \mathbf{XUX}')\mathbf{F}' = \mathbf{FWF}'.$$

Note: Above it seems that we need to assume that $\mathbf{W}$ is nnd.

Using now the fact that, on account of (5), $\mathbf{X} = \mathbf{WF}'\mathbf{B}$ for some $q \times p$ matrix $\mathbf{B}$, the equation (15) may be written in the form

$$\begin{aligned}\mathbf{CFWF}' &= \mathbf{WF}'\mathbf{B}[\mathbf{B}'\mathbf{FW}'\mathbf{F}'(\mathbf{FWF}')^-\mathbf{FWF}'\mathbf{B}]^-\mathbf{B}'\mathbf{FW}'\mathbf{F}' \\ &= \mathbf{WF}'\mathbf{B}[\mathbf{B}' \cdot \mathbf{FWF}'(\mathbf{FWF}')^-\mathbf{FWF}' \cdot \mathbf{B}]^-\mathbf{B}'\mathbf{FWF}' \\ &= \mathbf{WF}'\mathbf{B}(\mathbf{B}'\mathbf{FWF}'\mathbf{B})^-\mathbf{B}'\mathbf{FWF}' \\ &= \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{B}'\mathbf{FWF}', \end{aligned} \tag{16}$$

where we have used the symmetry of $\mathbf{W}$ and the equality $\mathbf{X}'\mathbf{W}^-\mathbf{X} = \mathbf{B}'\mathbf{F}\mathbf{W}\mathbf{F}'\mathbf{B}$. Since $\mathbf{W}$ is nonnegative definite, we can cancel, in view of the rank cancellation rule, the right-most $\mathbf{F}'$ from each side of

$$\mathbf{C}\mathbf{F}\mathbf{W}\mathbf{F}' = \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{B}'\mathbf{F}\mathbf{W}\mathbf{F}', \tag{17}$$

which then becomes (7), i.e., (16) is equivalent to (7), thus showing that the sets of BLUEs of $\mathbf{X}\boldsymbol{\beta}$ in models $\mathscr{M}$ and $\mathscr{M}_F$ coincide.

So far we have shown that the following equivalent claims guarantee that $\mathbf{Fy}$ is linearly sufficient for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ and then each BLUE of $\mathbf{X}\boldsymbol{\beta}$ under the transformed model $\mathscr{M}_F$ is the BLUE of $\mathbf{X}\boldsymbol{\beta}$ under the original model $\mathscr{M}$ and vice versa:

(a) $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{W}\mathbf{F}')$,

(c) $\operatorname{rank}(\mathbf{X} : \mathbf{V}\mathbf{F}') = \operatorname{rank}(\mathbf{W}\mathbf{F}')$.

It remains to prove that (a) [and thereby (c)] is equivalent to each of the following:

(b) $\mathscr{N}(\mathbf{F}) \cap \mathscr{C}(\mathbf{X} : \mathbf{V}) \subset \mathscr{C}(\mathbf{V}\mathbf{X}^\perp)$,

(d) $\mathscr{C}(\mathbf{X}'\mathbf{F}') = \mathscr{C}(\mathbf{X}')$ and $\mathscr{C}(\mathbf{F}\mathbf{X}) \cap \mathscr{C}(\mathbf{F}\mathbf{V}\mathbf{X}^\perp) = \{\mathbf{0}\}$,

(e) the best linear predictor of $\mathbf{y}$ based on $\mathbf{Fy}$, $\operatorname{BLP}(\mathbf{y}; \mathbf{Fy})$, is almost surely equal to a linear function of $\mathbf{Fy}$ which does not depend on $\boldsymbol{\beta}$.

Let us follow the steps of the proof of Baksalary & Kala (1986, Thm. 1, p. 333) who considered the linear sufficiency of an estimable $\mathbf{K}\boldsymbol{\beta}$. They show that $\mathbf{Fy}$ is linearly sufficient for $\mathbf{K}\boldsymbol{\beta}$ under $\mathscr{M}$ if and only if

$$\mathscr{N}(\mathbf{F}\mathbf{X} : \mathbf{F}\mathbf{V}\mathbf{M}) \subset \mathscr{N}(\mathbf{K} : \mathbf{0}), \tag{BK-86a}$$

which in case of $\mathbf{K} = \mathbf{X}$ becomes

$$\mathscr{N}(\mathbf{F}\mathbf{X} : \mathbf{F}\mathbf{V}\mathbf{M}) \subset \mathscr{N}(\mathbf{X} : \mathbf{0}). \tag{BK-86b}$$

The proof is simple. Now $\mathbf{Fy}$ is linearly sufficient for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ if and only if there exists a matrix $\mathbf{A}$ such that

$$\mathbf{A}(\mathbf{F}\mathbf{X} : \mathbf{F}\mathbf{V}\mathbf{M}) = (\mathbf{X} : \mathbf{0}). \tag{18}$$

In general,

$$\begin{aligned}\text{there exists } \mathbf{A} \text{ such that } \ \mathbf{A}\mathbf{Z} = \mathbf{Y},\ \text{i.e., } \mathbf{Z}'\mathbf{A}' = \mathbf{Y}' &\iff \mathscr{C}(\mathbf{Y}') \subset \mathscr{C}(\mathbf{Z}') \\ \iff \mathscr{C}(\mathbf{Z}')^\perp \subset \mathscr{C}(\mathbf{Y}')^\perp &\iff \mathscr{N}(\mathbf{Z}) \subset \mathscr{N}(\mathbf{Y}).\end{aligned} \tag{19}$$

Using (19) we see that (18) has a solution for $\mathbf{A}$ if and only if (BK86b) holds.

Perhaps the condition (BK86b) could be added to the list (a), … , (e) ?

Baksalary & Kala (1986) also show that if (BK-86a) holds, then every representation of the BLUE of $\mathbf{K}\boldsymbol{\beta}$ in the induced model $\mathscr{M}_F$ is also the BLUE of $\mathbf{K}\boldsymbol{\beta}$ in the original model $\mathscr{M}$.

Anyways, our task is to show that the solvability of (18) [or equivalently the condition (BK86b)] implies (b) etc and vice versa.

Equation (18) has a solution for $\mathbf{A}$ if and only if

$$\mathscr{C}\begin{pmatrix}\mathbf{X}'\\ \mathbf{0}\end{pmatrix} \subset \mathscr{C}\begin{pmatrix}\mathbf{X}'\mathbf{F}'\\ \mathbf{MVF}'\end{pmatrix}. \tag{20}$$

Now (20) implies (please confirm!) that $\mathscr{C}(\mathbf{X}') = \mathscr{C}(\mathbf{X}'\mathbf{F}')$, and hence (20) implies

$$\mathscr{C}\begin{pmatrix}\mathbf{X}'\mathbf{F}'\\ \mathbf{0}\end{pmatrix} \subset \mathscr{C}\begin{pmatrix}\mathbf{X}'\mathbf{F}'\\ \mathbf{MVF}'\end{pmatrix}. \tag{21}$$

In view of Section 5.12 (p. 130), (21) holds if and only if

$$\mathscr{C}(\mathbf{FX}) \cap \mathscr{C}(\mathbf{FVX}^{\perp}) = \{\mathbf{0}\}. \tag{22}$$

Hence we have shown that (20) implies (d). The reverse implication is obvious because $\mathscr{C}(\mathbf{X}') = \mathscr{C}(\mathbf{X}'\mathbf{F}')$ and (22) together certainly imply (20). Condition (d) appears in Baksalary & Kala (1986, Corollary 2, p. 334).

How about (20) and (b)? We can proceed corresponding to Isotalo & Puntanen (2006a, p. 1018) in their proof concerning the linear prediction sufficiency. So, let us first write (20) as

$$\mathscr{C}\left\{\left[\begin{pmatrix}\mathbf{X}'\\ \mathbf{MV}\end{pmatrix}\mathbf{F}'\right]^{\perp}\right\} \subset \mathscr{C}\left[\begin{pmatrix}\mathbf{X}'\\ \mathbf{0}\end{pmatrix}^{\perp}\right]. \tag{23}$$

The inclusion (23) implies

$$\mathscr{C}\left\{(\mathbf{X} : \mathbf{VM})\left[\begin{pmatrix}\mathbf{X}'\\ \mathbf{MV}\end{pmatrix}\mathbf{F}'\right]^{\perp}\right\} \subset \mathscr{C}\left[(\mathbf{X} : \mathbf{VM})\ \begin{pmatrix}\mathbf{X}'\\ \mathbf{0}\end{pmatrix}^{\perp}\right]. \tag{24}$$

In light of Prop. 5.7 (p. 137), the following holds:

$$\mathscr{C}(\mathbf{A}) \cap \mathscr{C}(\mathbf{B}) = \mathscr{C}[\mathbf{B}(\mathbf{B}'\mathbf{A}^{\perp})^{\perp}], \quad \mathscr{N}(\mathbf{A}') \cap \mathscr{C}(\mathbf{B}) = \mathscr{C}[\mathbf{B}(\mathbf{B}'\mathbf{A})^{\perp}].$$

Hence the left-hand side of (24) becomes

$$\mathscr{N}(\mathbf{F}') \cap \mathscr{C}(\mathbf{X} : \mathbf{VM}).$$

The right-hand side of (24) is obviously $\mathbf{VM}$; this can be seen from the fact

$$\begin{pmatrix}\mathbf{I}_p - \mathbf{P}_{\mathbf{X}'} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_n\end{pmatrix} \in \left\{\begin{pmatrix}\mathbf{X}'\\ \mathbf{0}\end{pmatrix}^{\perp}\right\}.$$

Thus we have shown that (20) implies (b).

To prove that (b), i.e., $\mathscr{N}(\mathbf{F}) \cap \mathscr{C}(\mathbf{X} : \mathbf{VM}) \subset \mathscr{C}(\mathbf{VM})$, implies the linear sufficiency of $\mathbf{Fy}$, we let $\mathbf{S}$ be such a matrix that $\mathbf{Sy}$ is the BLUE for $\mathbf{X}\boldsymbol{\beta}$. This implies that $\mathbf{SVM} = \mathbf{0}$, i.e., $\mathscr{C}(\mathbf{VM}) \subset \mathscr{N}(\mathbf{S})$, so that (b) implies the inclusion

$$\mathscr{N}(\mathbf{F}) \cap \mathscr{C}(\mathbf{X} : \mathbf{VM}) \subset \mathscr{C}(\mathbf{VM}) \subset \mathscr{N}(\mathbf{S}),$$

from which we immediately get

$$\mathscr{N}(\mathbf{F}) \cap \mathscr{C}(\mathbf{X} : \mathbf{VM}) \subset \mathscr{N}(\mathbf{S}) \cap \mathscr{C}(\mathbf{X} : V\mathbf{M}).$$

Denoting $\mathbf{N} = (\mathbf{X} : \mathbf{VM})$, the above inclusion can be equivalently expressed as

$$\mathscr{C}(\mathbf{F}' : \mathbf{N}^{\perp})^{\perp} \subset \mathscr{C}(\mathbf{S}' : \mathbf{N}^{\perp})^{\perp},$$

i.e.,

$$\mathscr{C}(\mathbf{S}' : \mathbf{N}^{\perp}) \subset \mathscr{C}(\mathbf{F}' : \mathbf{N}^{\perp}). \tag{25}$$

"Premultiplying" (25) by $\mathbf{N}'$ yields $\mathscr{C}(\mathbf{N}'\mathbf{S}') \subset \mathscr{C}(\mathbf{N}'\mathbf{F}')$, and hence there exists a matrix $\mathbf{B}$ such that

$$\mathbf{N}'\mathbf{S}' = \mathbf{N}'\mathbf{F}'\mathbf{B}', \quad \text{i.e.,} \quad \mathbf{BF}(\mathbf{X} : \mathbf{VM}) = \mathbf{S}(\mathbf{X} : \mathbf{VM}) = (\mathbf{X} : \mathbf{0}).$$

Thus we have finally shown that (b) implies that $\mathbf{Fy}$ is linearly sufficient for $\mathbf{X}\boldsymbol{\beta}$.

We skip the proof of (e) but let it it be mentioned that in view of Prop. 9.2 (p. 203), the BLP of $\mathbf{y}$ based on $\mathbf{Fy}$ is

$$\begin{aligned} \mathrm{BLP}(\mathbf{y}; \mathbf{Fy}) &= \boldsymbol{\mu}_{\mathbf{y}} + \mathrm{cov}(\mathbf{y}, \mathbf{Fy})[\mathrm{cov}(\mathbf{Fy})]^{-}[\mathbf{Fy} - \mathrm{E}(\mathbf{Fy})] \\ &= \mathbf{X}\boldsymbol{\beta} + \mathbf{VF}'(\mathbf{FVF}')^{-}\mathbf{F}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \\ &= [\mathbf{I}_n - \mathbf{VF}'(\mathbf{FVF}')^{-}\mathbf{F}]\mathbf{X}\boldsymbol{\beta} + \mathbf{VF}'(\mathbf{FVF}')^{-}\mathbf{Fy}\,, \end{aligned} \tag{26}$$

which means that (e) can be expressed as

$$\mathbf{VF}'(\mathbf{FVF}')^{-}\mathbf{FX} = \mathbf{X}. \tag{27}$$

Here we may mention the references Müller (1987), Drygas (1983), Sengupta & Jammalamadaka (2003, Ch. 11), and Isotalo & Puntanen (2006b). □