15.10 Exercises: Some Solutions (November 26, 2011)

15.1. Prove Proposition 15.5 (p. 340).

• Solution to Ex. 15.1:

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15.2. Prove Proposition 15.12 (p. 351).

15.3. Prove (see page 352):

$$\tilde{\boldsymbol\beta} - \tilde{\boldsymbol\beta}_{(i)} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{i}_i \,\frac{\dot\varepsilon_i}{\dot m_{ii}}\,, \quad \frac{\dot\varepsilon_i}{\dot m_{ii}} = \tilde\delta = \mathrm{BLUE}(\delta \mid \mathscr{M}_Z)\,.$$

15.4. Consider the models defined as in (15.144) (p. 351). Show that the F-test statistics for testing $\delta = 0$ under $\mathscr{M}_Z$ is

$$t_i^2(\mathbf{V}) = \frac{\dot\varepsilon_i^2}{\dot m_{ii}\tilde\sigma_{(i)}^2} = \frac{\tilde\delta_i^2}{\widetilde{\mathrm{var}}(\tilde\delta_i)}\,,$$

where $\dot{\boldsymbol\varepsilon} = \dot{\mathbf{M}}\mathbf{y} = \mathbf{V}^{-1}\tilde{\boldsymbol\varepsilon}$, $\tilde{\boldsymbol\varepsilon}$ is the BLUE's residual, and $(n-p-1)\tilde\sigma_{(i)}^2 = \mathrm{SSE}_{(i)}(\mathbf{V}) = \mathrm{SSE}_Z(\mathbf{V})$. The statistics $t_i^2(\mathbf{V})$ is a generalized version of the externally Studentized residual, cf. (8.192) (p. 182).

15.5. Suppose that $\mathbf{V}$ has the intraclass correlation structure and $\mathbf{1} \in \mathscr{C}(\mathbf{X})$, where $\mathbf{X}_{n\times p}$ has full column rank. Show that then $\mathrm{DFBETA}_i(\mathbf{V}) = \mathrm{DFBETA}_i(\mathbf{I})$. What about the equality between Cook's distance COOK_i^2 and the generalized Cook's distance $\mathrm{COOK}_i^2(\mathbf{V})$?

15.6. Express the Cook's distance $\mathrm{COOK}_i^2(\mathbf{V})$ as a function of an appropriate Mahalanobis distance.

15.7. Consider a partitioned model $\mathscr{M}_{12} = \{\mathbf{y},\, \mathbf{X}\boldsymbol\beta,\, \mathbf{V}\}$, where $\mathbf{X}$ has full column rank and $\mathbf{V}$ is positive definite. Suppose that $\hat{\boldsymbol\beta}_1$ is fully efficient under the small model $\mathscr{M}_1 = \{\mathbf{y},\, \mathbf{X}_1\boldsymbol\beta_1,\, \mathbf{V}\}$. Show that

$$\mathrm{eff}(\hat{\boldsymbol\beta}_1 \mid \mathscr{M}_{12}) = 1 \iff \mathbf{X}_1'\mathbf{X}_2\tilde{\boldsymbol\beta}_2(\mathscr{M}_{12}) = \mathbf{X}_1'\mathbf{X}_2\hat{\boldsymbol\beta}_2(\mathscr{M}_{12})\,.$$

Isotalo, Puntanen & Styan (2007).

15.8. Consider the models $\mathscr{M}$ and $\mathscr{M}_{(i)}$, defined in (15.144) (p. 351), where $\mathbf{X}$ and $(\mathbf{X} : \mathbf{i}_i)$ have full column ranks. Suppose that $\hat{\boldsymbol\beta} = \tilde{\boldsymbol\beta}$ under $\mathscr{M}$.

Show that the equality $\hat{\boldsymbol{\beta}}_{(i)} = \tilde{\boldsymbol{\beta}}_{(i)}$ for all $i = 1, \dots, n$ holds if and only if $\mathbf{MVM} = c^2\mathbf{M}$ for some nonzero $c \in \mathbb{R}$.

Nurhonen & Puntanen (1992a), Isotalo, Puntanen & Styan (2007).

15.9. According to (8.192) (p. 182), the F-test statistic for testing the hypothesis $\delta = 0$ under the model $\mathscr{M}_Z(\mathbf{I}) = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta} + \mathbf{u}\delta, \sigma^2\mathbf{I}\}$, where $\mathbf{u} = \mathbf{i}_i$, becomes

$$t_i^2 = \frac{\mathbf{y}'\mathbf{P}_{\mathbf{Mu}}\mathbf{y}}{\frac{1}{n-p-1}\mathbf{y}'(\mathbf{M} - \mathbf{P}_{\mathbf{Mu}})\mathbf{y}} = \frac{\hat{\varepsilon}_i^2}{\frac{1}{n-p-1}\,\mathrm{SSE}_{(i)}\, m_{ii}}\,,$$

which is the squared externally Studentized residual. Under the model $\mathscr{M}_Z(\mathbf{I})$ the test statistic t_i^2 follows an F-distribution with 1 and $n-p-1$ degrees of freedom. Denote $\mathscr{M}_Z(\mathbf{V}) = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta} + \mathbf{u}\delta, \sigma^2\mathbf{V}\}$. Prove that under the model $\mathscr{M}_Z(\mathbf{V})$: $t_i^2 \sim \mathrm{F}(1, n-p-1) \iff \mathbf{MVM} = c^2\mathbf{M}$ for some $c \neq 0$.

Nurhonen & Puntanen (1991), Rao & Mitra (1971b, Ch. 9).

15.10. Write up the solution to Exercise 15.9 when $\mathbf{Z} = (\mathbf{1} : \mathbf{i}_n)$. Moreover, confirm that the following statements are equivalent:

(a) $\mathbf{CVC} = c^2\mathbf{C}$ for some $c \neq 0$,

(b) $\mathbf{V} = \alpha^2\mathbf{I} + \mathbf{a}\mathbf{1}' + \mathbf{1}\mathbf{a}'$, where $\mathbf{a}$ is an arbitrary vector and α is any scalar ensuring the positive definiteness of $\mathbf{V}$.

Above $\mathbf{C}$ denotes the centering matrix. Confirm also that the eigenvalues of $\mathbf{V}$ in (b) are $\alpha^2 + \mathbf{1}'\mathbf{a} \pm \sqrt{n\mathbf{a}'\mathbf{a}}$, each with multiplicity one, and α^2 with multiplicity $n-2$. These results appear useful when studying the robustness of the Grubbs's test for detecting a univariate outlier.

Baksalary & Puntanen (1990b), Baksalary, Nurhonen & Puntanen (1992), Lehman & Young (1993), Markiewicz (2001).

• SOLUTION TO EX. 15.10:

Consider first the equation

$$\mathbf{CVC} = c^2\mathbf{C}\,, \quad \text{where } c \neq 0\,. \tag{1}$$

Our task is is to characterize all positive definite matrices $\mathbf{V}$ which satisfy (1). Baksalary & Puntanen (1990b) proceed by using Theorem 2 of Baksalary (1984), which gives the following general positive definite solution:

$$\mathbf{V} = c^2\mathbf{C} + \mathbf{T}_1\mathbf{w}\mathbf{1}' + \mathbf{1}\mathbf{w}'\mathbf{T}_1' + \nu\mathbf{1}\mathbf{1}', \tag{2}$$

where ν is an arbitrary scalar and $\mathbf{w} \in \mathbb{R}^{n-1}$ is an arbitrary vector satisfying the condition

$$\mathbf{w}'\mathbf{w} < c^2\nu\,.$$

Here, as in the solution of Exercise 3.10 (p. 104),

$$\mathbf{T} = (\mathbf{t}_1 : \ldots : \mathbf{t}_{n-1} : \mathbf{t}_n) = (\mathbf{T}_1 : \tfrac{1}{\sqrt{n}}\mathbf{1}_n)\,,$$

where the vectors $\mathbf{t}_i$ are the orthonormal eigenvectors of $\mathbf{C}$ and $\mathbf{C}$ has the eigenvalue decomposition

$$\mathbf{C} = \mathbf{T\Lambda T}' = \mathbf{T}\begin{pmatrix} \mathbf{I}_{n-1} & \mathbf{0} \\ \mathbf{0}' & 0 \end{pmatrix}\mathbf{T}' = \mathbf{T}_1\mathbf{T}_1'\,.$$

It is interesting to rewrite (1) as

$$\mathbf{C}(c^{-2}\mathbf{V})\mathbf{C} = \mathbf{C}\,, \quad \text{i.e.,} \quad c^{-2}\mathbf{V} \in \{\mathbf{C}^-_{\text{pd}}\}\,.$$

In view of the solution to Exercise 3.10, the general expression for $c^{-2}\mathbf{V} \in \{\mathbf{C}^-_{\text{pd}}\}$ is

$$c^{-2}\mathbf{V} = \mathbf{C} + \mathbf{1f}'\mathbf{T}_1' + \mathbf{T}_1\mathbf{f1}' + \delta\,\mathbf{11}', \tag{3}$$

for any $\mathbf{f} \in \mathbb{R}^{n-1}$ and $\delta \in \mathbb{R}$ which satisfy

$$\mathbf{f}'\mathbf{f} < \delta\,. \tag{4}$$

Multiplying (3) by c^2 gives the general pd expression for $\mathbf{V}$:

$$\begin{aligned}\mathbf{V} &= c^2\mathbf{C} + c^2\mathbf{1f}'\mathbf{T}_1' + c^2\mathbf{T}_1\mathbf{f1}' + c^2\delta\,\mathbf{11}' \\ &:= c^2\mathbf{C} + \mathbf{1w}'\mathbf{T}_1' + \mathbf{T}_1\mathbf{w1}' + c^2\delta\,\mathbf{11}',\end{aligned} \tag{5}$$

which is of the form (2).

Baksalary & Puntanen (1990b) show that the sets specified by (b) and (2) are equal. Let's copy part their proof here.

Because the columns of $(\mathbf{T}_1 : \mathbf{1})$ span the entire $\mathbb{R}^n$, it is clear that $\mathbf{a}$ in (b) can be uniquely decomposed as $\mathbf{a} = \mathbf{T}_1\mathbf{a}_1 + \gamma\mathbf{1}$ for some $\mathbf{a}_1 \in \mathbb{R}^{n-1}$ and $\gamma \in \mathbb{R}$, and hence

$$\mathbf{T}_1\mathbf{a}_1 = \mathbf{a} - \gamma\mathbf{1}\,,$$
$$\mathbf{T}_1\mathbf{a}_1\mathbf{1}' = \mathbf{a1}' - \gamma\mathbf{11}', \quad \mathbf{1a}_1'\mathbf{T}_1' = \mathbf{1a}' - \gamma\mathbf{11}'.$$

Choosing $\mathbf{w} = \mathbf{a}_1$ in (b) yields

$$\begin{aligned}\mathbf{V} &= c^2\mathbf{C} + \mathbf{a1}' + \mathbf{1a}' - 2\gamma\mathbf{11}' + \nu\mathbf{11}' \\ &= c^2\mathbf{C} + \mathbf{a1}' + \mathbf{1a}' + (\nu - 2\gamma)\mathbf{11}' \\ &= c^2(\mathbf{I} - \mathbf{J}) + \mathbf{a1}' + \mathbf{1a}' + n(\nu - 2\gamma)\mathbf{J}\,.\end{aligned}$$

Substituting then $c^2 = \alpha^2$ and $\nu = 2\gamma + \alpha^2/n$ into (2) yields (b). This shows that the class (2) includes all matrices having the structure (b); (b) is called the class of *Baldessari's structure*. The converse inclusion follows similarly, by substituting $\alpha^2 = c^2$ and $\mathbf{a} = \mathbf{T}_1\mathbf{a}_1 + \gamma\mathbf{1}$, where $\gamma = \nu/2 - c^2/(2n)$.

How to confirm that the eigenvalues of

$$\mathbf{V} = \alpha^2\mathbf{I} + \mathbf{a}\mathbf{1}' + \mathbf{1}\mathbf{a}' := \alpha^2\mathbf{I} + \mathbf{W}$$

are $\alpha^2 + \mathbf{1}'\mathbf{a} \pm \sqrt{n\mathbf{a}'\mathbf{a}}$, each with multiplicity one, and α^2 with multiplicity $n-2$? We observe that

$$\operatorname{ch}(\mathbf{V}) = \{\alpha^2 + \operatorname{ch}(\mathbf{a}\mathbf{1}' + \mathbf{1}\mathbf{a}')\}\,. \tag{c}$$

If $\operatorname{rk}(\mathbf{1} : \mathbf{a}) = 2$, then $\mathbf{q} \in \mathscr{C}(\mathbf{1} : \mathbf{a})^\perp$ as stated in the solution of Exercise 3.10, and thereby 0 is an eigenvalue of $\mathbf{W}$ of multiplicity 2. The remaining two eigenvalues of $\mathbf{W} = \mathbf{a}\mathbf{1}' + \mathbf{1}\mathbf{a}'$ are $\mathbf{1}'\mathbf{a} \pm \sqrt{n\mathbf{a}'\mathbf{a}}$. We skip the proof.

Some references; those not appearing in the Tricks References, written in full:

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