18.11 Exercises: Some Solutions (February 8, 2012)

18.1. Let $\mathbf{A}_{2\times 2}$ be symmetric. Show that

$$\text{(a) } \mathbf{A} \geq_{\mathsf{L}} \mathbf{0} \iff \text{(b) } \operatorname{tr}(\mathbf{A}) \geq 0 \text{ and } \det(\mathbf{A}) \geq 0 .$$

• SOLUTION TO EX. 18.1:

Proof of (a) $\implies$ (b):
$\mathbf{A} \geq_{\mathsf{L}} \mathbf{0} \implies \lambda_1 \geq 0, \lambda_2 \geq 0 \implies \operatorname{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 \geq 0$, $\det(\mathbf{A}) = \lambda_1\lambda_2 \geq 0$.

Proof of (b) $\implies$ (a):
$\det(\mathbf{A}) = \lambda_1\lambda_2 \geq 0 \implies \lambda_i \geq 0$, for $i = 1, 2$ or $\lambda_i \leq 0$, for $i = 1, 2$. Using $\operatorname{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 \geq 0$ forces $\lambda_i \geq 0$, for $i = 1, 2$. □

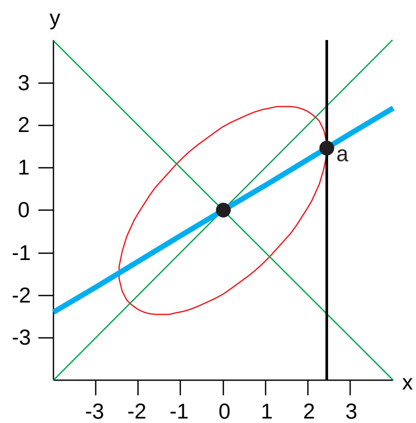

Figure 18.1 (See Exercise 18.6) A 95% confidence region for the observations from $\mathrm{N}_2(\mathbf{0}, \mathbf{\Sigma})$.

18.2 (This is Ex. 18.6). The points inside the ellipse

$$\mathcal{A} = \{\, \mathbf{z} \in \mathbb{R}^2 \colon (\mathbf{z} - \boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{z} - \boldsymbol{\mu}) = \chi^2_{\alpha,2} \,\}$$

form a $100(1-\alpha)\%$ confidence region for the observations from $\mathrm{N}_2(\boldsymbol{\mu}, \mathbf{\Sigma})$; see Figure 18.1. Assume that we have 10 000 observations from $\mathbf{z}$ and that $\boldsymbol{\mu} = \mathbf{0}$, $\mathbf{\Sigma} = \left(\begin{smallmatrix} 1 & \varrho \\ \varrho & 1 \end{smallmatrix}\right)$.

(a) What is your guess for the regression line when y is explained by x (and the constant).
(b) Find the vector $\mathbf{a} = \binom{a_1}{a_2}$ in Figure 18.1 when $\alpha = 0.05$. $[a_1^2 = \sigma^2 \chi^2_{0.05,2} = \chi^2_{0.05,2}]$
(c) What is the cosine of the angle between the regression line and the major axis.
(d) Why is $a_1 > 1.96$?

• SOLUTION TO EX. 18.6

The vector $\mathbf{a} = \binom{a_1}{a_2}$ satisfies the equation of the ellipse $\mathcal{A}$ and it must lie also on the regression line whose equation is now $y = \varrho x$:

$$a_2 = \varrho a_1 \,, \qquad (a_1, a_2)' \mathbf{\Sigma}^{-1} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \chi^2_{\alpha,\,2} \,.$$

Suppose that $\sigma_x^2 = \sigma_y^2 = \sigma^2 (= 1$ in Figure 18.1), so that

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} \sigma^2 & \sigma^2 \varrho \\ \sigma^2 \varrho & \sigma^2 \end{pmatrix}^{-1} = \frac{1}{\sigma^2(1-\varrho^2)} \begin{pmatrix} 1 & -\varrho \\ -\varrho & 1 \end{pmatrix} .$$

Substituting $a_2 = \varrho a_1$ into $\mathcal{A}$:

$$\begin{aligned} (a_1, \varrho a_1)' \mathbf{\Sigma}^{-1} \begin{pmatrix} a_1 \\ \varrho a_1 \end{pmatrix} &= \frac{a_1^2}{\sigma^2(1-\varrho^2)} (1, \varrho)' \begin{pmatrix} 1 & -\varrho \\ -\varrho & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \varrho \end{pmatrix} \\ &= \frac{a_1^2}{\sigma^2(1-\varrho^2)} (1 + \varrho^2 - 2\varrho^2) \\ &= \frac{1}{\sigma^2(1-\varrho^2)} (1-\varrho^2) = \frac{a_1^2}{\sigma^2} \\ &= \chi^2_{\alpha,\,2} \,, \end{aligned}$$

from which a_1 is

$$a_1 = \sigma \sqrt{\chi^2_{\alpha,\,2}} = \sigma 2.45 \quad \text{if } \alpha = 0.05.$$

Notice that 95% of the probability mass of the distribution of x lies in the interval $(-1.96\sigma, 1.96\sigma)$. This interval is shorter than $(-a_1, a_1) = (-2.45\sigma, 2.45\sigma)$

The eigenvalues of $\mathbf{\Sigma}$ are $\sigma^2(1+\varrho)$ ja $\sigma^2(1-\varrho)$ and the corresponding eigenvectors are

$$\mathbf{t}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad \mathbf{t}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

If $\varrho > 0$, then the slope of the first principle axis is 1. The cosine of the angle between the regression line and the first principle axis is

$$\cos\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\\varrho\end{pmatrix}\right) = \frac{1+\varrho}{\sqrt{2(1+\varrho^2)}} \,.$$
